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TOPICAL REVIEW

Continuous symmetries of difference equations

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Abstract
Lie group theory was originally created more than 100 years ago as a tool for solving ordinary and partial differential equations. In this article we review the results of a much more recent program: the use of Lie groups to study difference equations. We show that the mismatch between continuous symmetries and discrete equations can be resolved in at least two manners. One is to use generalized symmetries acting on solutions of difference equations, but leaving the lattice invariant. The other is to restrict them to point symmetries, but to allow them to also transform the lattice.

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Contents

1. Introduction

The symmetry theory of differential equations is well understood. It goes back to the classical work of Sophus Lie and is reviewed in numerous modern books and articles [12, 16, 20, 22, 37, 48, 69, 87, 112, 118, 122–125, 133, 211, 218, 230, 247–249, 262]. As a matter of fact, Lie group theory is now a very general and useful tool for obtaining exact analytic solutions of large classes of differential equations, specially nonlinear ones.

The application of Lie group theory to discrete equations is much more recent and a vigorous development of the theory only started in the 1990s [10, 15, 32, 33, 41, 51, 59–61, 63–68, 75, 76, 92, 98, 101–107, 119, 120, 126, 129, 131, 135–137, 140–142, 144–147, 151–162, 165–178, 183–188, 193, 206, 221–223, 229, 231, 235–238, 240, 246, 257, 260, 263–267].

The purpose of this article is to provide a review of the progress made.

In this whole field of research one uses group theory to do for difference equations what has been done for differential ones. This includes generating new solutions from old ones, identifying equations that can be transformed into each other, performing symmetry reduction and identifying integrable equations.

When adapting the group theoretical approach from differential equations to difference ones, we must answer three basic questions:

1. What do we mean by symmetries?
2. How do we find the symmetries of a difference system?
3. What do we do with the symmetries once we know them?

1.1. Lie groups and differential equations

Let us first briefly review the situation for differential equations.

Let us consider a general system of differential equations

$$E_a(x, u, u_x, u_{2x}, \ldots, u_{nx}) = 0, \qquad x \in \mathbb{R}^p, u \in \mathbb{R}^q, a = 1, \ldots, N, \tag{1.1}$$

where u_{nx} denotes all (partial) derivatives of u of order n. The numbers p, q, n and N are all nonnegative integers.

We are interested in the symmetry group $\mathfrak{G}$ of system (1.1), i.e. in the local Lie group of local point transformations taking solutions of equation (1.1) into solutions of the same equation. Point transformations in the space $X \times U$ of independent and dependent variables have the form

$$\tilde{x} = \Lambda_\lambda(x, u), \qquad \tilde{u} = \Omega_\lambda(x, u), \tag{1.2}$$

where λ denotes the group parameters. Thus

$$\Lambda_0(x, u) = x, \qquad \Omega_0(x, u) = u,$$

and the inverse transformation $(\tilde{x}, \tilde{u}) \mapsto (x, u)$ exists, at least locally.

The transformations (1.2) of local coordinates in $X \times U$ also determine the transformations of functions $u = f(x)$ and of derivatives of functions. A group $\mathfrak{G}$ of local point transformations of $X \times U$ will be a symmetry group of system (1.1) if the fact that $u(x)$ is a solution implies that $\tilde{u}(\tilde{x})$ is also a solution.

How does one find the symmetry group $\mathfrak{G}$? Instead of looking for 'global' transformations as in equation (1.2) one looks for infinitesimal ones, i.e. one looks for the Lie algebra $\mathfrak{g}$ that corresponds to $\mathfrak{G}$. A one-parameter group of infinitesimal point transformations will have the form

$$\begin{aligned} &\tilde{x}_i = x_i + \lambda\xi_i(x, u), \qquad \tilde{u}_\alpha = u_\alpha + \lambda\phi_\alpha(x, u), \\ &|\lambda| \ll 1 \quad 1 \leqslant i \leqslant p, \quad 1 \leqslant \alpha \leqslant q. \end{aligned} \tag{1.3}$$

The search for the symmetry algebra $\mathfrak{g}$ of a system of differential equations is best formulated in terms of vector fields acting on the space $X \times U$ of independent and dependent variables. Indeed, consider the vector field

$$\hat{X} = \sum_{i=1}^{p} \xi_i(x, u)\partial_{x_i} + \sum_{\alpha=1}^{q} \phi_\alpha(x, u)\partial_{u_\alpha}, \tag{1.4}$$

where the coefficients ξ_i and ϕ_α are the same as in equation (1.3). If these functions are known, the vector field (1.4) can be integrated to obtain the finite transformations (1.2). Indeed, all we have to do is to integrate the equations

$$\frac{\mathrm{d}\tilde{x}_i}{\mathrm{d}\lambda} = \xi_i(\tilde{x}, \tilde{u}), \qquad \frac{\mathrm{d}\tilde{u}_\alpha}{\mathrm{d}\lambda} = \phi_\alpha(\tilde{x}, \tilde{u}), \tag{1.5}$$

subject to the initial conditions

$$\tilde{x}_i|_{\lambda=0} = x_i \qquad \tilde{u}_\alpha|_{\lambda=0} = u_\alpha. \tag{1.6}$$

This provides us with a one-parameter group of local Lie point transformations of the form (1.2) where λ is the group parameter.

The vector field (1.4) tells us how the variables x and u transform. We also need to know how derivatives such as $u_x, u_{xx}, \ldots$ transform. This is given by the prolongation of the vector field $\hat{X}$.

We have

$$\mathrm{pr}\,\hat{X} = \hat{X} + \sum_\alpha \left\{ \sum_i \phi_\alpha^{x_i} \partial_{u_{\alpha,x_i}} + \sum_{i,k} \phi_\alpha^{x_i x_k} \partial_{u_{\alpha,x_i x_k}} + \sum_{i,k,l}^{x_i x_k x_l} \phi_\alpha^{x_i x_k x_l} \partial_{u_{\alpha,x_i x_k x_l}} + \cdots \right\}, \tag{1.7}$$

where the coefficients in the prolongation can be calculated recursively, using the total derivative operator,

$$D_{x_i} = \partial_{x_i} + u_{\alpha,x_i}\partial_{u_\alpha} + u_{\alpha,x_a x_i}\partial_{u_{\alpha,x_a}} + u_{\alpha,x_a x_b x_i}\partial_{u_{\alpha,x_a x_b}} + \cdots \tag{1.8}$$

(a summation over repeated indices is to be understood). The recursive formulae are

$$\begin{aligned} &\phi_\alpha^{x_i} = D_{x_i}\phi_\alpha - (D_{x_i}\xi_a)u_{\alpha,x_a}, \qquad \phi_\alpha^{x_i x_k} = D_{x_k}\phi_\alpha^{x_i} - (D_{x_k}\xi_a)u_{\alpha,x_i x_a}, \\ &\phi_\alpha^{x_i x_k x_l} = D_{x_l}\phi_\alpha^{x_i x_k} - (D_{x_l}\xi_a)u_{\alpha,x_i x_k x_a}, \end{aligned} \tag{1.9}$$

etc.

The invariance condition for system (1.1) is expressed in terms of the operator (1.7) as

$$\mathrm{pr}^{(n)}\hat{X}E_a|_{E_1=\cdots=E_N=0} = 0, \qquad a = 1, \ldots, N, \tag{1.10}$$

where $\mathrm{pr}^{(n)}\hat{X}$ is the prolongation (1.7) calculated up to order n (where n is the order of system (1.1)).

Equation (1.10) provides a system of linear partial differential equations for the functions $\xi_i(x,u)$ and $\phi_\alpha(x,u)$, in which the variables x and u figure as independent variables. By definition of point transformations the coefficients ξ_i and ϕ_α depend only on $(x_1,\ldots,x_p,u_1,\ldots,u_q)$, not on any derivative of u_α. The action of $pr^{(n)}\hat{X}$ in equation (1.10) will, on the other hand, introduce terms in (1.10), involving the derivatives $\frac{\partial^k u}{\partial x_1^{k_1}\ldots\partial x_p^{k_p}}$, $k = k_1+\cdots+k_p$, $1 \leqslant k \leqslant n$. We use equations (1.1) to eliminate N (the number of equations) such derivatives. We then collect all linearly independent remaining expressions in the derivatives and set the coefficients of these expressions equal to zero. This provides the 'determining equations': a set of linear partial differential equations for the functions $\xi_i(x,u)$ and $\phi_\alpha(x,u)$. The order of the system of determining equations is the same as the order of the studied system (1.1); however, the determining system is linear, even if the system (1.1) is nonlinear. It is usually overdetermined and not difficult to solve. Computer programs using various symbolic languages exist that derive the determining system and solve it, or at least partially solve it [16, 42, 100, 243, 247].

The solution of the determining system may be trivial, i.e. $\xi_i = 0, \phi_\alpha = 0$. Then the symmetry approach is of no avail. Alternatively, the general solution may depend on a finite number K of integration constants. The Lie algebra of the symmetry group, the 'symmetry algebra', for short, is then K-dimensional and must be identified as an abstract Lie algebra [99, 128, 198, 227]. Finally, the general solution of the determining equations may involve arbitrary functions and the symmetry algebra is infinite-dimensional. For instance, for a linear PDE the linear superposition principle is reflected by the presence in the Lie algebra of an operator depending on the general solution of the studied equation. In turn, this general solution depends on arbitrary functions, e.g. the Cauchy data.

So far we have considered only point transformations, as in equation (1.2), in which the new variables $\tilde{x}$ and $\tilde{u}$ depend only on the old ones, x and u. More general transformations are 'contact transformations', where $\tilde{x}$ and $\tilde{u}$ also depend on first derivatives of u [13, 22, 118, 122, 211, 249]. A still more general class of transformations are generalized transformations, also called 'Lie-Bäcklund' transformations [13, 132, 211]. In principle these involve derivatives of arbitrary orders.

When studying generalized symmetries, and sometimes also point symmetries, it is convenient to use a different formalism, namely that of evolutionary vector fields. Let us first consider the case of Lie point symmetries, i.e. vector fields of the form (1.4) and their prolongations (1.7). With each vector field (1.4) we can associate its evolutionary counterpart $\hat{X}_e$, defined as

$$\hat{X}_e = Q_\alpha(x,u,u_x)\partial_{u_\alpha}, \qquad Q_\alpha = \phi_\alpha - \xi_j u_{\alpha,x_j}. \tag{1.11}$$

The prolongation of the evolutionary vector field (1.11) is defined as

$$\begin{aligned} &\mathrm{pr}\,\hat{X}_e = Q_\alpha\partial_{u_a} + Q_\alpha^{x_j}\partial_{u_{\alpha,x_j}} + Q_\alpha^{x_jx_k}\partial_{u_{\alpha,x_jx_k}} + \cdots \\ &Q_\alpha^{x_j} = D_{x_j}Q_\alpha, \qquad Q_\alpha^{x_jx_k} = D_{x_j}D_{x_k}Q_\alpha, \ldots. \end{aligned} \tag{1.12}$$

The functions Q_α are called the characteristics of the vector field. Observe that $\hat{X}_e$ and $\mathrm{pr}\hat{X}_e$ do not act on the independent variables x_j. For Lie point symmetries evolutionary and ordinary vector fields are entirely equivalent and it is easy to pass from one to the other. Indeed, equation (1.11) gives the connection between the two. The symmetry algorithms for calculating the symmetry algebra $\mathfrak{g}$ in terms of ordinary, or evolutionary vector fields, are also equivalent. Equation (1.10) is simply replaced by

$$\mathrm{pr}^{(n)}\hat{X}_e E_a|_{E_1=\cdots=E_N=0} = 0, \qquad a = 1, \ldots, N. \tag{1.13}$$

The reason that equations (1.10) and (1.13) are equivalent is the following:

$$\mathrm{pr}^{(n)}\hat{X}_e = \mathrm{pr}^{(n)}X - \xi_i D_i. \tag{1.14}$$

The total derivative D_i acts like a generalized symmetry of equation (1.1), i.e.,

$$D_i E_a|_{E_1=E_2=\cdots=E_N=0} = 0 \qquad i = 1, \ldots, p, \quad a = 1, \ldots, N. \tag{1.15}$$

Equations (1.14) and (1.15) prove that systems (1.10) and (1.13) are equivalent. Equation (1.15) itself follows from the fact that $D_i E_a = 0$ is a differential consequence of equation (1.1); hence, every solution of equation (1.1) is also a solution of equation (1.15) (i.e. the action of D_i on solutions is trivial).

To find generalized symmetries of order k, we use equation (1.11) but allow the characteristics Q_α to depend on all derivatives of u up to order k. The prolongation is calculated using equation (1.12). The symmetry algorithm is again equation (1.13).

A very useful property of evolutionary symmetries is that the functions Q_α provide compatible flows. This means that the system of equations

$$\frac{\partial u_\alpha}{\partial \lambda} = Q_\alpha \tag{1.16}$$

is compatible with system (1.1) when $u_\alpha = u_\alpha(x, \lambda)$. In particular, group-invariant solutions, i.e., solutions invariant under a subgroup of $\mathfrak{G}$, are obtained as fixed points

$$Q_\alpha = 0. \tag{1.17}$$

If Q_α is the characteristic of a point transformation, then (1.17) is a system of quasilinear first-order partial differential equations. They can be solved and their solutions can be substituted into (1.1), yielding the invariant solutions explicitly. We mention that there is no guarantee that equation (1.16) or even (1.17) will have physically meaningful solutions.

Many different extensions of Lie's original method of group invariant solutions exist. Among them we mention, first of all, *conditional symmetries* [21, 80, 81, 164]. For differential equations, they were introduced under several different names [21, 50, 164, 215] in order to obtain dimensional reductions of partial differential equations, beyond those obtained by using ordinary Lie symmetries.

Another valuable extension is the concept of *partial symmetries*. They correspond to the existence of a subset of solutions which, without necessarily being invariant, are mapped into each other by the transformation [47, 49]. Further extensions are given by *asymptotic symmetries* [88, 90], when extra symmetries are obtained in the asymptotic regime, or *approximate symmetries* [14, 82, 125] where one considers the symmetries of approximate solutions of a system depending on a small parameter.

1.2. Lie groups and difference schemes

Let us now return to the problem at hand, namely symmetries of difference systems. We wish to study the continuous symmetries and use Lie algebra techniques. However, the equations are now discrete, i.e. they involve functions $u(x)$ that are themselves continuous, but evaluated, or sampled, at discrete points. Several different approaches to this problem have been developed and will be discussed below.

In any approach we must take into account that we are dealing with two objects. One is the difference equation itself. Since $u(x)$ is not necessarily a scalar, this may actually be a system of equations. For simplicity, unless otherwise stated, we will restrict to scalar equations. The other object is the lattice, which may be *a priori* given and not subject to group transformations. Alternatively, the considered Lie group $\mathfrak{G}$ may act on solutions and on the lattices. In order to allow a unified treatment of different approaches, we shall give the difference scheme by a system of equations, involving the variables x evaluated at K different points, with $2 \leqslant K < \infty$. The equations will have the form

$$\begin{aligned} &E_a\left(\{x_k\}_{k=n+M}^{n+N}, \{u_k\}_{k=n+M}^{n+N}\right) = 0, \qquad a = 1, \ldots, n_E \quad x \in \mathbb{R}^p, \quad u \in \mathbb{R} \\ &K = N - M + 1, \quad n, M, N \in \mathbb{Z}, \quad N > M, \quad u_k \equiv u(x_k). \end{aligned} \tag{1.18}$$

The number of points involved is not necessarily the same in all equations.

Let us first discuss the case of an ordinary difference scheme (OΔS), when we have just one independent variable x and one dependent scalar, $u(x)$. In this case the number of equations n_E in (1.18) must be $n_E = 2$ and these two equations must satisfy the independence conditions

$$\frac{\partial(E_1, E_2)}{\partial(x_{n+N}, u_{n+N})} \neq 0, \qquad \frac{\partial(E_1, E_2)}{\partial(x_{n+M}, u_{n+M})} \neq 0. \tag{1.19}$$

These nondegeneracy conditions make it possible to calculate (x_{n+N}, u_{n+N}), or (x_{n+M}, u_{n+M}), respectively, if all the other points are given. In the continuous limit the spacings between any two neighbouring points go to zero. One of the equations (1.18) reduces to an ordinary differential equation (ODE) of order $K' \leqslant K - 1$ (e.g. for $K = 3$ we obtain a second-, or first-order ODE). The other equation, or some combination of the two equations (1.18) in the continuum limit reduces to an identity (like $0 = 0$).

Thus, an OΔS corresponds to two relations between K points x_k on a line and the values $u_k = u(x_k)$ at these points. The points are not necessarily equally spaced. The symmetry group $\mathfrak{G}$ of the system (1.18) will act by point transformations, as in equation (1.2). The prolonged action will act on all points of the lattice, in particular on all points figuring in equation (1.18). The Lie algebra $\mathfrak{g}$ that corresponds to the group $\mathfrak{G}$ is realized by vector fields of the form (1.4), just as in the case of differential equations, and equation (1.5) also holds. The prolongation of the vector field (1.7) is different. For the OΔS (1.18) the prolongation is

$$\mathrm{pr}\,\hat{X} = \sum_{k=n+M}^{n+N} \xi(x_k, u_k)\partial_{x_k} + \sum_{k=n+M}^{n+N} \phi(x_k, u_k)\partial_{u_k}. \tag{1.20}$$

i.e. we sum over all points of the lattice that figure in equation (1.18).

As an example of an OΔS consider a three-point scheme approximating a second-order ODE on a uniform lattice. We put

$$E_1 = \frac{u_{n+1} - 2u_n + u_{n-1}}{(x_{n+1} - x_n)^2} - u_n^2 = 0; \qquad E_2 = x_{n+1} - 2x_n + x_{n-1} = 0. \tag{1.21}$$

The continuous limit is

$$E_1 = 0 \rightarrow u_{xx} - u^2 = 0, \qquad E_2 = 0 \rightarrow 0 = 0. \tag{1.22}$$

The case of partial difference equations (PΔS) is similar, but somewhat more complicated. Let us restrict to the simplest nontrivial case, namely one scalar function of two independent variables $u(x, t)$. The lattice is a distribution of points in a plane and we will call these points $P_{m,n}$ and their coordinates $(x_{m,n}, t_{m,n})$. Here x and t are Cartesian coordinates, or some other coordinates of the points $P_{m,n}$. The integers m, n label points of the lattice. A solution of the difference system will provide us with expressions for x, t and u as functions of m, n and some arbitrary functions of one discrete variable (either m, n or some function of m and n). An example is given below in equation (1.24). In order to allow the group transformations to

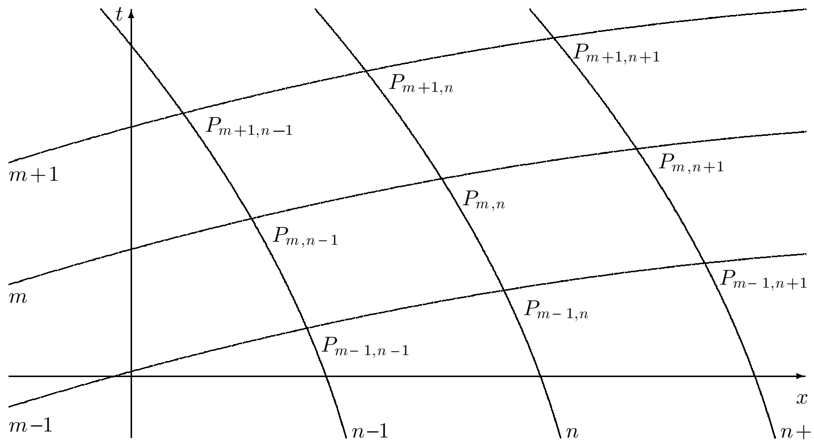

Figure 1. Example of a two-dimensional lattice obtained as a solution of the system (1.18).

act on the lattices we again describe the PΔS by the relation (1.18). The minimal number of equations n_E is 3, and we then have three coupled functional equations for three continuous functions x, t and u of two discrete labels m and n (see figure 1). The solution of the system will depend on a certain number of arbitrary functions of one variable. The actual number of these functions depends on the number of points used in the scheme. These functions can then be determined from a combination of initial and boundary conditions, just as in the case of partial differential equations.

As an example of a system of equations (1.18) with $n_E = 3$, consider a discretization of the PDE $u_{xt} = 0$, namely

$$\begin{aligned} E_1 &= \frac{1}{t_{m+1,n} - t_{m,n}} \left\{ \frac{u_{m+1,n+1} - u_{m+1,n}}{x_{m+1,n+1} - x_{m+1,n}} - \frac{u_{m,n+1} - u_{m,n}}{x_{m,n+1} - x_{m,n}} \right\} = 0 \\ E_2 &= t_{m,n+1} - t_{m,n} = 0, \qquad E_3 = x_{m+1,n} - x_{m,n} = 0, \end{aligned} \tag{1.23}$$

where $u_{m,n} = u(x_{m,n}, t_{m,n})$. The general solution of this system involves four arbitrary functions of one variable, namely

$$t_{m,n} = \alpha(m), \qquad x_{m,n} = \beta(n), \qquad u_{m,n} = f(x_{m,n}) + g(t_{m,n}). \tag{1.24}$$

An alternative approach to symmetries of PΔS [160] is to choose $n_E = 5$ in the system (1.18) (for three variables x, t and u). The system is then overdetermined and certain compatibility conditions must be satisfied. The additional equations will further specify the lattice and remove some, or all of the arbitrary functions obtained when imposing only three equations ($n_E = 3$). When we impose $n_E > 3$ equations (1.18) on x, t, u, then $n_E - 3$ of them will play the role of initial, or boundary conditions for the first three equations.

In the case of two independent variables x and t and one dependent one u we can write the symmetry vector field $\hat{X}_{m,n}$ as

$$\hat{X}_{m,n} = \xi_{n,m}(x_{n,m}, t_{n,m}, u_{n,m})\partial_{x_{n,m}} + \tau_{n,m}(x_{n,m}, t_{n,m}, u_{n,m})\partial_{t_{n,m}} + \phi_{n,m}(x_{n,m}, t_{n,m}, u_{n,m})\partial_{u_{n,m}}, \tag{1.25}$$

and its prolongation as

$$\mathrm{pr}\,\hat{X} = \sum_{k,l} \hat{X}_{m+k,n+l} \tag{1.26}$$

where the summation is over all points figuring in the system (1.18).

The invariance condition for the system (1.18) is the same as in the continuous case, namely (1.10).

More generally, for a system with k independent variables $\vec{x}$ we introduce a fixed coordinate frame. Each point of the lattice will be labelled by k integers, e.g. $P_{n_1,n_2,\dots,n_k}$ and correspondingly will have coordinates $\vec{x}_{n_1,\dots,n_k}$. At least $k+1$ equations (1.18) are needed to define a scalar equation and the lattice.

The formalism is somewhat heavy but allows us to consider quite general point transformations. The actual lattice emerges, together with the function $u(\vec{x})$, as a solution of the system (1.18). A standard, equally spaced orthogonal (Cartesian) lattice is the special case in which the equations $E_2, \dots, E_k$ in (1.18) are 'one point' equations $x_i = n_i$. In this special case no continuous transformation of the independent variables is allowed.

The reason for introducing this general formalism is precisely to be able to consider continuous symmetries such as rotations, Lorentz transformations, Galilei transformations, etc as point symmetries for systems on the lattices.

It is to be stressed that in this approach the lattice is obtained as a part of the solution of the difference system. Lie point transformations will in general transform a solution given on one lattice into a solution on a different lattice.

1.3. Outline of the review

In section 2 we discuss *point symmetries* of difference equations defined on fixed, nontransforming lattices [33, 92, 98, 119, 120, 135–137, 140, 165, 168, 173, 184–187, 193, 222, 223, 236, 238]. The symmetry transformations are assumed to have the form (1.2). They must take solutions into solutions and the lattice into itself. This approach is fruitful mainly for differential–difference equations (DΔE's), where not only the dependent variables, but also some of the independent ones are continuous.

Section 3 is devoted to *generalized point symmetries* on fixed lattices [75, 76, 105, 145–147, 156, 157, 162]. The concept of symmetry is generalized in that transformations act simultaneously at several points of the lattice, possibly infinitely many ones. In the continuous limit they reduce to point transformations. This approach is fruitful mainly for linear equations, or equations that can be linearized by a transformation of variables.

In section 4 we consider *generalized symmetries* on fixed lattices [101–103, 107, 144, 245, 246]. This approach is fruitful for discrete nonlinear integrable equations, i.e., nonlinear difference equations possessing a Lax pair. The symmetries are generalized ones, treated in the evolutionary formalism. In a continuous limit they reduce to point and generalized symmetries of integrable differential equations.

Point symmetries transforming solutions and lattices [15, 32, 59–68, 158–160, 174] are considered in section 5. The transformations have the form (1.2) and they act on solutions and on lattices simultaneously. The lattices themselves are given by difference equations and their form is dictated by the symmetries. Their main application is to discretize given differential equations while preserving their symmetries.

A brief conclusion with an overview of the results presented and pending issues is given in section 6.

2. Point symmetries of difference equations defined on fixed, nontransforming lattices

The essentially continuous techniques for finding Lie symmetries for differential equations can be extended in a natural way to the discrete case by acting just on the continuous variables [165, 168, 184–188, 222, 223], leaving the lattice invariant. Transformations of the lattice are considered only at the level of the group which itself is finite or discrete. Depending on the discrete equation we are considering we can have translations on the lattice by multiples of the lattice spacing and rotations through fixed angles, if the lattice transforms into itself under such rotation.

In section 2.1 we consider, as an example, the Lie point symmetries of the discrete time Toda lattice [144]. In section 2.2, we review the steps necessary to obtain continuous symmetries for differential–difference equations (DΔE's) and then in section 2.3 we carry out the explicit calculations in the case of the Toda equation in 1+1 dimensions. In doing so we will present and compare the results contained in our articles on the subject, which correspond to both the *intrinsic method* and the *differential equation method* proposed earlier [166–169], and those of the *global method* introduced by Quispel *et al* [222].

Apart from the standard Toda lattice we will consider a Toda lattice equation with variable coefficients and will show how, by analysing its symmetry group, one can find the Lie point transformation which maps it into the standard Toda lattice equation with constant coefficients. In section 2.4, we will present some results on the classification of nonlinear DΔE's with nearest-neighbour interactions. Finally, symmetries of a two-dimensional Toda lattice are discussed in section 2.5.

2.1. Lie point symmetries of the discrete time Toda equation

The discrete time Toda equation [114] is one of the integrable completely discrete partial differential equations (ΔΔ E) [9, 109, 113, 115–117, 148, 203, 204, 207, 216, 239, 241, 242, 250] and is given by

$$\Delta_{\text{Toda}} = \mathrm{e}^{u_{n,m}-u_{n,m+1}} - \mathrm{e}^{u_{n,m+1}-u_{n,m+2}} - \alpha^2(\mathrm{e}^{u_{n-1,m+2}-u_{n,m+1}} - \mathrm{e}^{u_{n,m+1}-u_{n+1,m}}) = 0. \tag{2.1}$$

Defining

$$t = m\sigma_t, \qquad v_n(t) = u_{n,m}, \qquad \alpha = \sigma_t \tag{2.2}$$

we find that equation (2.1) reduces to the continuous-time Toda equation:

$$\Delta^{(2)}_{\text{Toda}} = v_{n,tt} - \mathrm{e}^{v_{n-1}-v_n} + \mathrm{e}^{v_n-v_{n+1}} = 0, \tag{2.3}$$

when $\sigma_t \to 0$ and $m \to \infty$ in such a way that t remains finite. The Toda equation (2.3) is probably the best known and most studied differential–difference equation [253, 254]. It plays, in the case of lattice equations, the same role as the Korteweg–de Vries equation for partial differential equations. It was obtained by Toda [252] in order to explain the Fermi, Pasta and Ulam results [77] on the numerical experiments on the equipartition of energy in a nonlinear lattice of interacting oscillators. As shown below in section 4.2.2 equation (2.3) reduces, in the continuum limit, to the potential Korteweg–de Vries equation. It can be encountered in many applications from solid state physics to DNA biology, from molecular chain dynamics to chemistry [36, 228, 253].

Equation (2.1) can be obtained as the compatibility condition of the following over-determined pair of linear difference equations:

$$\psi_{n-1,m} + \left(\alpha + \frac{1}{\alpha} - \alpha \mathrm{e}^{u_{n-1,m+1}-u_{n,m}} - \frac{\mathrm{e}^{u_{n,m}-u_{n,n+1}}}{\alpha}\right)\psi_{n,m} + \mathrm{e}^{u_{n,m}-u_{n+1,m}}\psi_{n+1,m} = \lambda\psi_{n,m} \tag{2.4}$$

$$\psi_{n,m+1} = \psi_{n,m} - \alpha\,\mathrm{e}^{u_{n,m+1}-u_{n+1,m}}\psi_{n+1,m}. \tag{2.5}$$

Let us now determine the Lie point symmetries of the discrete time Toda lattice (2.1). We shall use the notation introduced in section 1.2. The difference scheme (1.18) in this case reduces to (2.1) and

$$x_{n,m} = n\sigma_x, \qquad t_{n,m} = m\sigma_t. \tag{2.6}$$

Equations (2.6) from now on will be denoted as $\Delta_{\text{Lattice}} = 0$.

A Lie point symmetry is defined by giving its infinitesimal generators, i.e. by the vector field (1.25). The invariance condition (1.10) then reads:

$$\mathrm{pr}\,\hat{X}\,\Delta_{\mathrm{Toda}}|_{(\Delta_{\mathrm{Toda}}=0,\,\Delta_{\mathrm{Lattice}}=0)} = 0, \qquad \mathrm{pr}\,\hat{X}\,\Delta_{\mathrm{Lattice}}|_{(\Delta_{\mathrm{Toda}}=0,\,\Delta_{\mathrm{Lattice}}=0)} = 0 \tag{2.7}$$

The action of (1.26) on the lattice equation (2.6) gives $\xi_{n,m} = 0$, $\tau_{n,m} = 0$ and thus the variables x and t are invariant. When we act with (1.26) on the Toda equation, we get

$$\begin{aligned} \mathrm{e}^{u_{n,m}-u_{n,m+1}}[\phi_{n,m} - \phi_{n,m+1}] - \mathrm{e}^{u_{n,m+1}-u_{n,m+2}}[\phi_{n,m+1} - \phi_{n,m+2}] \\ - \alpha^2\{\mathrm{e}^{u_{n-1,m+2}-u_{n,m+1}}[\phi_{n-1,m+2} - \phi_{n,m+1}] \\ - \mathrm{e}^{u_{n,m+1}-u_{n+1,m}}[\phi_{n,m+1} - \phi_{n+1,m}]\}|_{\Delta_{\mathrm{Toda}}=0} = 0 \end{aligned} \tag{2.8}$$

i.e. a functional equation for the function $\phi_{n,m}$. Taking into account the functional constraint provided by the discrete Toda lattice equation (2.1), the independent variables appearing in equation (2.8) are: $u_{n,m}$, $u_{n,m+1}$, $u_{n,m+2}$ and $u_{n+1,m}$. To solve equation (2.8) we differentiate it successively with respect to its independent variables up to the moment when we have an ODE. If we differentiate equation (2.8) twice with respect to $u_{n,m+2}$ we get that $\phi_{n,m} = c_1\mathrm{e}^{u_{n,m}} + c_2$, where c_1 and c_2 are two integration constants (that can depend on $x_{n,m}$ and $t_{n,m}$). Introducing this result into equation (2.8) we get that c_1 must be equal to zero. Taking into account that, due to the form of the lattice, all points are independent we get that c_2 must be just a constant.

To sum up, the discrete time Toda equation (2.1) considered on a fixed lattice has only a one-dimensional continuous symmetry group. It consists of the translation of the dependent variable u, i.e. $\tilde{u}_{n,m} = u_{n,m} + \lambda$ with λ constant. This symmetry is obvious from the beginning as equation (2.1) does not involve $u_{n,m}$ itself but only differences between values of u at different points of the lattice. Other transformations that leave the lattice and solutions invariant will be discrete [155]. In this case they are simply translations of x and t by integer multiples of the lattice spacing σ_x and σ_t.

More general symmetries are obtained if we relax the lattice conditions. Equation (2.6) can be viewed as solutions of the lattice equations

$$\begin{aligned} x_{n+1,m} - x_{n,m} &= \sigma_x, \qquad & t_{n+1,m} - t_{n,m} &= 0, \\ x_{n,m+1} - x_{n,m} &= 0, \qquad & t_{n,m+1} - t_{n,m} &= \sigma_t, \end{aligned} \tag{2.9}$$

satisfying the boundary-initial conditions

$$x_{0,0} = 0, \qquad t_{0,0} = 0. \tag{2.10}$$

If we drop the conditions (2.10) the solution of (2.9) is

$$x_{n,m} = \sigma_x n + x_0, \qquad t_{n,m} = \sigma_t m + t_0, \tag{2.11}$$

where x_0 and t_0 are integration constants (as well as initial x and t values). Acting with the prolongation (1.26) on (2.9), instead of (2.6) we obtain a more general result. Namely, the discrete time Toda lattice system (2.1), (2.9) is invariant under continuous translations of x and t, in addition to those of u:

$$\hat{X}_1 = \partial_u, \qquad \hat{X}_2 = \partial_x, \qquad \hat{X}_3 = \partial_t.$$

The same conclusion holds in the general case of difference equations on fixed lattices. Lie algebra techniques will provide transformations of the continuous dependent variables only, though the transformations can depend on the discrete independent variables.

In section 4, we will see that the situation is completely different when generalized symmetries are considered. In section 5 we will consider transforming lattices, which greatly increases the role of point symmetries.

2.2. Lie point symmetries of differential–difference equations

Let us now consider the case of differential–difference equations (DΔE). For notational simplicity, let us restrict ourselves to scalar DΔE for one real function $u(n,t)$ depending on one lattice variable n and one continuous real variable, t. Moreover, we will only be interested in DΔE containing up to second-order derivatives, as those are the ones of particular interest in applications to dynamical systems. We write such equations as:

$$\Delta_n^{(2)} \equiv \Delta\big(t, n, u_{n+k}|_{k=a_0}^{b_0}, u_{n+k,t}|_{k=a_1}^{b_1}, u_{n+k,tt}|_{k=a_2}^{b_2}\big) = 0 \qquad a_j \leqslant b_j \in \mathbb{Z}, \tag{2.12}$$

with $u_n \equiv u_n(t)$.

The lattice is assumed to be uniform, time independent and fixed, the continuous variable t is the same at all points of the lattice. Thus to equation (2.12) we add

$$x_n = n, \qquad t_n = t_{n+1} = t, \tag{2.13}$$

to obtain the corresponding system (1.18). Examples of such equations which will be considered in the following are the Toda lattice equation (2.3) and the inhomogeneous Toda lattice [150]

$$\begin{aligned}\tilde{\Delta}_n^{(2)} = w_{,tt}(n) - \frac{1}{2}w_{,t} + \frac{1}{4} - \frac{n}{2} + \left[\frac{1}{4}(n-1)^2 + 1\right]\mathrm{e}^{w(n-1)-w(n)} \\ - \left[\frac{1}{4}n^2 + 1\right]\mathrm{e}^{w(n)-w(n+1)} = 0.\end{aligned} \tag{2.14}$$

We are interested in Lie point transformations which leave the solution set of equations (2.12) and (2.13) invariant. They have the form:

$$\tilde{t} = \Lambda_g(t, n, u_n(t)), \qquad \tilde{u}_{\tilde{n}}(\tilde{t}) = \Omega_g(t, n, u_n(t)), \qquad \tilde{n} = n \tag{2.15}$$

where g represents a set of continuous or discrete group parameters.

Continuous transformations of the form (2.15) are generated by a Lie algebra of vector fields of the form:

$$\widehat{X} = \tau_n(t, u_n(t))\partial_t + \phi_n(t, u_n(t))\partial_{u_n} \tag{2.16}$$

where n is treated as a discrete variable and we have set $\tilde{n} = n$, when considering continuous transformations.

Invariance of the condition (2.13) implies that τ does not depend on n.

As in the case of purely differential equations, the following invariance condition

$$\mathrm{pr}^{(2)}\widehat{X}\Delta_n^{(2)}\big|_{\Delta_n^{(2)}=0} = 0 \tag{2.17}$$

must be true if $\widehat{X}$ is to belong to the Lie symmetry algebra of $\Delta_n^{(2)}$. The symbol $\mathrm{pr}^{(2)}\widehat{X}$ denotes the second prolongation of the vector field $\widehat{X}$, i.e.

$$\begin{aligned}\mathrm{pr}^{(2)}\widehat{X} = \tau(t, u_n)\partial_t + \sum_{k=n-a}^{n+b}\phi_k(t, u_k)\partial_{u_k} + \sum_{k=n-a_1}^{n+b_1}\phi_k^t(t, u_k, u_{k,t})\partial_{u_{k,t}} \\ + \sum_{k=n-a_2}^{n+b_2}\phi_k^{tt}(t, u_k, u_{k,t}, u_{k,tt})\partial_{u_{k,tt}}\end{aligned} \tag{2.18}$$

with

$$\phi_k^t(t, u_k, u_{k,t}) = D_t\phi_k(t, u_k) - [D_t\tau(t, u_k)]u_{k,t} \tag{2.19}$$

$$\phi_k^{tt}(t, u_k, u_{k,t}, u_{k,tt}) = D_t\phi_k^t(t, u_k, u_{k,t}) - [D_t\tau(t, u_k)]u_{k,tt} \tag{2.20}$$

where D_t is the total derivative with respect to t.

Let us note moreover that ϕ_k^t and ϕ_k^{tt} are the prolongation coefficients with respect to the continuous variable. The prolongation with respect to the discrete variable is reflected in the summation over k.

Equation (2.17) is to be viewed as just one equation with n as a discrete variable; thus we have a finite algorithm for obtaining the determining equations, a usually overdetermined system of linear partial differential equations for $\tau(t, u_n)$ and $\phi_n(t, u_n)$.

A different approach consists of considering equation (2.12) as a system of coupled differential equations for the functions $u_n(t)$. Thus, in general we have infinitely many equations for infinitely many functions. In this case the Ansatz for the vector field $\widehat{X}$ would be:

$$\widehat{X} = \tau(t, \{u_j(t)\}_j)\partial_t + \sum_k \phi_k(t, \{u_j(t)\}_j)\partial_{u_k(t)} \tag{2.21}$$

where by $\{u_j(t)\}_j$ we mean the set of all $u_j(t)$ and j and k vary *a priori* over an infinite range. Calculating the second prolongation $\mathrm{pr}^{(2)}\widehat{X}$ in a standard manner (see equations (1.7) and (1.9)) and imposing

$$\mathrm{pr}^{(2)}\widehat{X}\Delta_n^{(2)}\big|_{\Delta_j^{(2)}=0} = 0 \quad \forall n, j \tag{2.22}$$

we obtain, in general, an infinite system of determining equations for an infinite number of functions. Conceptually speaking, this second method, called the *differential equation method* in [168], may give rise to a larger symmetry group than the *intrinsic method* we introduced before. In fact the intrinsic method yields purely point transformations, while the differential equation method can yield generalized symmetries with respect to the differences (but not the derivatives). In practice, it turns out that usually no higher order symmetries with respect to the discrete variable exist; then the two methods give the same result and the intrinsic method is simpler.

A third approach [222, 223] consists of interpreting the variable n as a continuous variable and consequently the DΔE as a differential-delay equation. In such an approach $u_{n+k}(t) \equiv \exp\left[\frac{k\partial}{\partial n}\right]\{u_n(t)\}$ and consequently the DΔE is interpreted as a partial differential equation of infinite order. In such a case formula (2.17) is meaningless as we are not able to construct the infinite order prolongation of a vector field $\widehat{X}$. The Lie symmetries are obtained by requiring that the solution set of the equation $\Delta_n^{(2)} = 0$ (2.3) be invariant under the infinitesimal transformation

$$\tilde{t} = t + \epsilon\tau(n, t, u_n(t)), \qquad \tilde{n} = n + \epsilon\nu(n, t, u_n(t)), \qquad \tilde{u}_{\tilde{n}}(\tilde{t}) = u_n(t) + \epsilon\phi(n, t, u_n(t)). \tag{2.23}$$

To obtain conditional symmetries for difference equations we add to equation (2.12) a constraining equation which we choose in such a way that it is automatically annihilated on its solution set by the prolongation of the vector field. Such an equation is the invariant surface condition

$$\Delta_n^{(1)} = \phi_n(t, u_n(t)) - \tau(t, u_n(t))u_{n,t}(t) = 0. \tag{2.24}$$

In general equations (2.12) and (2.24) may not be compatible; if they are, then equation (2.24) provides a reduction of the number of the independent variables by one. This is the essence of the symmetry reduction by conditional symmetries. Due to the fact that equation (2.24) is written in terms of τ and ϕ_n, which are the coefficients of the vector field $\widehat{X}$, the determining equations are nonlinear. The obtained vector fields do not form an algebra, nor even a vector space, since each vector field is adapted to a different condition [21, 50, 80, 164].

2.3. Symmetries of the Toda lattice

Let us now apply the techniques introduced in section 2.2 to the case of equation (2.3) with the lattice conditions (2.13). In this case equation (2.17) reduces to an overdetermined system of determining equations obtained by equating to zero the coefficients of $[v_{n,t}]^k$, $k = 0, 1, 2, 3$ and of $v_{n\pm 1}$. They imply:

$$\tau = at + d, \qquad \phi = b + 2an + ct, \qquad a, b, c, d \text{ real constants}, \tag{2.25}$$

corresponding to a four-dimensional Lie algebra generated by the vector fields:

$$\widehat{D} = t\partial_t + 2n\partial_{v_n}, \qquad \widehat{T} = \partial_t, \qquad \widehat{W} = t\partial_{v_n}, \qquad \widehat{U} = \partial_{v_n}. \tag{2.26}$$

The group transformation which will leave equation (2.3) invariant is hence:

$$\tilde{v}_n(\tilde{t}) = v_n(\tilde{t}\,\mathrm{e}^{-\lambda_4/2} - \lambda_3) + \lambda_2(\tilde{t}\,\mathrm{e}^{-\lambda_4/2} - \lambda_3) + \lambda_4 n + \lambda_1 \tag{2.27}$$

where λ_j, $j = 1, 2, 3, 4$, are real group parameters.

To the transformation (2.27) we can add some discrete ones [168]:

$$\tilde{n} = n + N \quad N \in \mathbb{Z} \tag{2.28}$$

and

$$(t, v_n) \to (-t, v_n); \qquad (t, v_n) \to (t, -v_{-n}). \tag{2.29}$$

We write the symmetry group of equation (2.3) as a semidirect product:

$$\mathfrak{G} = \mathfrak{G}_D \otimes \mathfrak{G}_C \tag{2.30}$$

where $\mathfrak{G}_D$ are the discrete transformations (2.28), (2.29) and the invariant subgroup $\mathfrak{G}_C$ corresponds to the transformation (2.27).

In fact, if we complement the Lie algebra (2.26) by the vector field

$$\widehat{Z} = \frac{\partial}{\partial n} \tag{2.31}$$

and require, at the end of the calculations, that the corresponding group parameter be integer, the commutation relations become:

$$[\widehat{Z}, \widehat{D}] = 2\widehat{U}; \qquad [\widehat{D}, \widehat{T}] = -\widehat{T}; \qquad [\widehat{D}, \widehat{W}] = \widehat{W}; \qquad [\widehat{T}, \widehat{W}] = \widehat{U}. \tag{2.32}$$

A complete classification of the one-dimensional subgroups of $\mathfrak{G}$ can easily be obtained [168]. The one-dimensional subalgebras are [166, 169]

$$\begin{array}{lllll} \{\widehat{Z} + a\widehat{D} + b\widehat{U}\}, & \{\widehat{Z} + a\widehat{T} + k\widehat{W}\}, & & \{\widehat{Z} + \epsilon\widehat{W}\}, & \{\widehat{Z} + a\widehat{U}\}, \\ \{\widehat{T} + c\widehat{W}\}, & \{\widehat{D} + c\widehat{U}\}, & \{\widehat{U}\}, & \{\widehat{Z}\}, & \{\widehat{W}\}, \\ (a, b, c) \in \mathbb{R}; & a \neq 0; & & k = 0, 1, -1; & \epsilon = \pm 1. \end{array} \tag{2.33}$$

Nontrivial solutions, corresponding to reductions with respect to *continuous subgroups* $\mathfrak{G}_0 \subset \mathfrak{G}_C$, are obtained by considering invariance of the Toda lattice under $\{\widehat{T} + c\widehat{W}\}$, or $\{\widehat{D} + c\widehat{U}\}$. They are:

$$v_n(t) = p - \frac{1}{2}ct^2 - \sum_{j=1}^{n} \log(q - cj), \tag{2.34}$$

$$v_n(t) = p + 2(n + c)\log(t) - \sum_{j=0}^{n} \log[q + (2c - 1)j + j^2], \tag{2.35}$$

where p and q are arbitrary integration parameters.

Reduction by the purely *discrete subgroup*, $\mathfrak{G}_0 \subset \mathfrak{G}_D$, given in equation (2.28) implies the invariance of equation (2.3) under discrete translations of n and makes it possible to impose the periodicity condition

$$u(n+N,t) = u(n,t). \tag{2.36}$$

This reduces the DΔE (2.3) to an ordinary differential equation (or a finite system of equations). For example for $N = 2$, we get the following reduction of the sinh-Gordon equation

$$v_{,tt} = -4\sinh v, \tag{2.37}$$

while for $N = 3$, we get a reduction of the ordinary Tzitzeica differential equation [258, 259]:

$$v_{,tt} = \mathrm{e}^{-2v} - \mathrm{e}^{v}. \tag{2.38}$$

Let us now consider a subgroup $\mathfrak{G}_0 \subset \mathfrak{G}$ that is not contained in $\mathfrak{G}_C$, nor in $\mathfrak{G}_D$, i.e. a *nonsplitting subgroup* of $\mathfrak{G}$ [219, 220, 262]. A reduction corresponding to $\widehat{Z}+a\widehat{D}+b\widehat{U}$ yields the equation

$$F''(y) = \mathrm{e}^{-b}\{\exp[F(y\,\mathrm{e}^{a}) - F(y) + a] - \exp[F(y) - F(y\,\mathrm{e}^{-a}) - a]\} \tag{2.39}$$

where the symmetry variables $F(y)$ and y are defined by:

$$y = t\,\mathrm{e}^{-an}, \qquad v_n(t) = an^2 + bn + F(y). \tag{2.40}$$

Using the subalgebra $\widehat{Z} + a\widehat{T} + k\widehat{W}$ we get

$$F''(y) = \mathrm{e}^{F(y+a)-F(y)} - \mathrm{e}^{F(y)-F(y-a)} - \frac{k}{a}, \tag{2.41}$$

where

$$y = t - an, \qquad v_n(t) = \frac{k}{2a}t^2 + F(y). \tag{2.42}$$

Equation (2.39) can be called a 'differential dilation' type equation; it involves one independent variable y, but the function F and its derivatives are evaluated at the point y and at the dilated points $y\,\mathrm{e}^{a}$ and $y\,\mathrm{e}^{-a}$. Equation (2.41) is a differential delay equation which has interesting solutions, such as the soliton and periodic solutions of the Toda lattice (for $k = 0$).

The other two nonsplitting subgroups give rise to linear delay equations which can be solved explicitly.

This same calculation can also be carried out for the inhomogeneous Toda lattice (2.14). The symmetry algebra is

$$\begin{aligned} &\widetilde{D} = 2\partial_{\tilde{t}} + \tfrac{1}{2}\partial_{w_n}, \qquad &&\widetilde{T} = \mathrm{e}^{-\tilde{t}/2}\big[\partial_{\tilde{t}} - \big(w_n - \tfrac{1}{2}\big)\partial_{w_n}\big] \\ &\widetilde{W} = 2\,\mathrm{e}^{\tilde{t}/2}\partial_{w_n}, \qquad &&\widetilde{U} = \partial_{w_n}. \end{aligned} \tag{2.43}$$

These vector fields have the same commutation relations as those of the Toda lattice (2.3). This is a necessary condition for the existence of a point transformation between the two equations. In fact by comparing the two sets of vector fields, we get the following transformation which transforms a solution $v_n(t)$ of equation (2.3) into a solution $w_n(\tilde{t})$ of equation (2.14)

$$\begin{aligned} &\tilde{t} = 2\log\left(\frac{t}{2}\right), \\ &w_n(\tilde{t}) = v_n(t) - 2\left(n - \frac{1}{2}\right)\log(t) + \sum_{j=0}^{n}\left[\frac{1}{4}(j-1)^2 + 1\right]. \end{aligned} \tag{2.44}$$

Let us now calculate the conditional symmetries of the Toda lattice. We assume that τ is not zero. The determining equation reads:

$$\begin{aligned}\phi_{n,tt} + \phi_{n,t}\phi_{n,v_n} + 2\phi_n\phi_{n,tv_n} + \phi_n\left[\phi_{n,v_n}^2 + \phi_n\phi_{n,v_nv_n)}\right] + [2\phi_n - \phi_{n-1} - \phi_{n+1}]\,\mathrm{e}^{v_n - v_{n+1}}\\ + [\phi_n - \phi_{n-1}][\phi_{n,t} - \phi_n\phi_{n,v_n}] = 0.\end{aligned} \tag{2.45}$$

This implies

$$\phi_n(t, v_n(t)) = \alpha(t) + \beta(t)n \tag{2.46}$$

with

$$\beta_{,tt} + \beta\beta_{,t} = 0, \qquad \alpha_{,tt} + \beta\alpha_{,t} = 0. \tag{2.47}$$

Solving equations (2.47) we get

$$\phi_n(t, v_n(t)) = \begin{cases} K_0 + \left(2\dfrac{K_1}{K_3} + nK_3\right)\tanh\left[\dfrac{K_3}{2}(t - t_0)\right], & \text{for } K_3 \neq 0 \\ \dfrac{K_0 t + K_1 + 2n}{t - t_0}, & \text{for } K_3 = 0. \end{cases} \tag{2.48}$$

For $K_3 = 0$ we get the results given in equation (2.25). For $K_3 \neq 0$ an additional 'symmetry' is given by $\phi = nK_3\tanh(K_3t/2)$. This gives a new explicit solution of the Toda lattice:

$$v_n(t) = u_0 + 2nK_3\log\left[\cosh\left(\frac{K_3}{2}t\right)\right]. \tag{2.49}$$

2.4. Classification of differential equations on a lattice

Group theoretical methods can also be used to classify equations according to their symmetry groups. This has been done in the case of partial differential equations [91] showing, for instance, that in the class of variable coefficient Korteweg–de Vries equations, the Korteweg–de Vries itself has the largest symmetry group. The same kind of results can also be obtained in the case of differential–difference equations.

Let us consider a class of equations involving nearest-neighbour interactions [173]

$$\Delta_n = u_{n,tt}(t) - F_n(t, u_{n-1}(t), u_n(t), u_{n+1}(t)) = 0, \tag{2.50}$$

where F_n is nonlinear in $u_k(t)$ and coupled, i.e. such that $F_{n,u_k} \neq 0$ for some $k \neq n$.

We consider point symmetries only; the continuous transformations of the form (2.15) are again generated by a Lie algebra of vector fields of the form (2.16). Taking into account the form of equation (2.50), we have

$$\widehat{X} = \tau(t)\partial_t + \left[\left(\frac{1}{2}\tau_{,t} + a_n\right)u_n + b_n(t)\right]\partial_{u_n} \tag{2.51}$$

with $a_{n,t} = 0$. The determining equations reduce to

$$\begin{aligned}&\frac{1}{2}\tau_{,ttt}u_n + b_{n,tt} + \left(a_n - \frac{3}{2}\tau_{,t}\right)F_n - \tau F_{n,t}\\ &\qquad - \sum_{k=0,\pm 1}\left[\left(\frac{1}{2}\tau_{,t} + a_{n+k}\right)u_{n+k} + b_{n+k}(t)\right]F_{n,u_{n+k}} = 0.\end{aligned} \tag{2.52}$$

Our aim is to solve equation (2.52) with respect to both the form of the nonlinear equation, i.e. F_n, and the symmetry vector field $\widehat{X}$, i.e. $(\tau(t), a_n, b_n(t))$. In other words, for every nonlinear interaction F_n we wish to find the corresponding maximal symmetry group $\mathfrak{G}$. Associated with any symmetry group $\mathfrak{G}$ there will be a whole class of nonlinear differential–difference equations related to each other by point transformations. To simplify the results, we will just look for the simplest element of a given class of nonlinear differential–difference equations, associated with a certain symmetry group. To do so we introduce so-called *allowed transformations*, i.e. a set of transformations of the form

$$\tilde{t} = \tilde{t}(t), \qquad \tilde{n} = n \qquad u_n(t) = \Omega_n(\tilde{u}_n(\tilde{t}), t) \tag{2.53}$$

that transform equation (2.50) into a different one of the same type. By a straightforward calculation we find that the only allowed transformations (2.53) are given by

$$\tilde{t} = \tilde{t}(t), \qquad \tilde{n} = n, \qquad u_n(t) = \frac{A_n}{\sqrt{\tilde{t}_{,t}(t)}}\tilde{u}_n(\tilde{t}) + B_n(t) \tag{2.54}$$

with $B_n(t)$, A_n, $\tilde{t}(t)$ arbitrary functions of their arguments.

Under an allowed transformation equation (2.50) is transformed into

$$\tilde{u}_{n,\tilde{t}\tilde{t}}(\tilde{t}) = \widetilde{F}_n(n, \tilde{t}, \tilde{u}_{n+1}(\tilde{t}), \tilde{u}_n(\tilde{t}), \tilde{u}_{n-1}(\tilde{t})) \tag{2.55}$$

with

$$\tilde{F}_n = \frac{1}{\tilde{t}_{,t}^{\frac{3}{2}} A_n}\left\{F_n(n, t, \{u_{n+1}(t), u_n(t), u_{n-1}(t)\}) - B_{n,tt} - \left[\frac{3}{4}\frac{\tilde{t}_{,tt}^2}{\tilde{t}_{,t}^{\frac{5}{2}}} - \frac{1}{2}\frac{\tilde{t}_{,ttt}}{\tilde{t}_{,t}^{\frac{3}{2}}}\right] A_n \tilde{u}_n(\tilde{t})\right\} \tag{2.56}$$

and a symmetry generator (2.51) into

$$\begin{aligned}\widehat{\tilde{X}} = [\tau(t)\tilde{t}_{,t}]\partial_{\tilde{t}} + \Bigg\{ &\left[\frac{\tau(t)}{2}\frac{\tilde{t}_{,tt}}{\tilde{t}_{,t}} + \frac{1}{2}\tau_{,t}(t) + a_n\right]\tilde{u}_n \\ &+ \frac{\tilde{t}_{,t}^{\frac{1}{2}}}{A_n}\left[-\tau(t)B_{n,t}(t) + B_n(t)\left(\frac{1}{2}\tau_{,t}(t) + a_n\right) + b_n(t)\right]\Bigg\}\partial_{\tilde{u}_n}.\end{aligned} \tag{2.57}$$

We see that, up to an allowed transformation, every one-dimensional symmetry algebra associated with equation (2.50) can be represented by one of the following vector fields:

$$\widehat{X}_1 = \partial_t + a_n^1 u_n \partial_{u_n} \qquad \widehat{X}_2 = a_n^2 u_n \partial_{u_n} \qquad \widehat{X}_3 = b_n(t)\partial_{u_n} \tag{2.58}$$

where a_n^j with $j = 1, 2$ are two arbitrary functions of n and $b_n(t)$ is an arbitrary function of n and t. The vector fields $\widehat{X}_j$, $j = 1, 2, 3$, are the symmetry vectors of the Lie point symmetries of the following nonlinear differential–difference equations:

$$\begin{aligned}&\widehat{X}_1: \quad u_{n,tt} = \mathrm{e}^{a_n^1 t} f_n(\xi_{n+1}, \xi_n, \xi_{n-1}), &&\text{with} \quad \xi_j = u_j\,\mathrm{e}^{-a_j^1 t} \\ &\widehat{X}_2: \quad u_{n,tt} = u_n f_n(t, \eta_{n+1}, \eta_{n-1}), &&\text{with} \quad \eta_j = \frac{(u_j)^{a_n^2}}{(u_n)^{a_j^2}} \\ &\widehat{X}_3: \quad u_{n,tt} = \frac{b_{n,tt}}{b_n}u_n + f_n(t, \zeta_{n+1}, \zeta_{n-1}) &&\text{with} \quad \zeta_j = u_j b_n(t) - u_n b_j(t).\end{aligned} \tag{2.59}$$

These equations are still quite general, as they are written in terms of arbitrary functions depending on three continuous variables. More specific equations are obtained for larger symmetry groups [173].

The Toda equation (2.3) is included in a class of equations whose infinitesimal symmetry generators satisfy a four-dimensional solvable symmetry algebra with a non-Abelian nilradical. The interactions in this class are given by

$$F_n(t, u_{n-1}(t), u_n(t), u_{n+1}(t)) = \mathrm{e}^{-2\frac{u_{n+1}-u_n}{\gamma_{n+1}-\gamma_n}} f_n(\xi) \tag{2.60}$$

where $\xi = [\gamma_n(t) - \gamma_{n+1}(t)]u_{n-1} + [\gamma_{n+1}(t) - \gamma_{n-1}(t)]u_n + [\gamma_{n-1}(t) - \gamma_n(t)]u_{n+1}$ and the function $\gamma_n(t)$ is such that $\gamma_{n+1}(t) \neq \gamma_n(t)$ and $\frac{\partial \gamma_n(t)}{\partial t} = 0$. The associated symmetry generators are

$$\widehat{X}_1 = \partial_{u_n}, \qquad \widehat{X}_2 = \partial_t, \qquad \widehat{X}_3 = t\partial_{u_n}, \qquad \widehat{Y} = t\partial_t + \gamma_n(t)\partial_{u_n}. \tag{2.61}$$

The Toda equation (2.3) is obtained by choosing $\gamma_n(t) = 2n$ and $f_n(\xi) = -1 + \mathrm{e}^{\frac{1}{2}\xi}$. Among the equations of the class (2.50), the Toda equation does not have the largest group of point symmetries.

A complete list of all equations of the type (2.50) with nontrivial symmetry group is given in the original article [173] with the additional assumption that the interaction and the vector fields depend continuously on n. Here we just give two examples of interactions with symmetry groups with dimension seven. The first one is solvable, nonnilpotent and its Lie algebra is given by

$$\begin{aligned} &\widehat{X}_1 = \partial_{u_n}, \qquad \widehat{X}_2 = (-1)^n\partial_{u_n}, \qquad \widehat{X}_3 = t\partial_{u_n}, \\ &\widehat{X}_4 = (-1)^n t\partial_{u-n}, \qquad \widehat{X}_5 = (-1)^n u_n\partial_{u_n}, \qquad \widehat{X}_6 = \partial_t, \qquad \widehat{X}_7 = t\partial_t + 2u_n\partial_{u_n}. \end{aligned} \tag{2.62}$$

The invariant equation is

$$u_{n,tt} = \frac{\gamma_n}{u_{n-1} - u_{n+1}}. \tag{2.63}$$

This algebra was not included in [173] because of its nonanalytical dependence on n (in $\hat{X}_2$, $\hat{X}_4$ and $\hat{X}_5$).

The second symmetry algebra is nonsolvable; it contains the simple Lie algebra $\mathrm{sl}(2, \mathbb{R})$ as a subalgebra. A basis of this algebra is

$$\begin{aligned} &\widehat{X}_1 = \partial_{u_n}, \qquad \widehat{X}_2 = t\partial_{u_n}, \qquad \widehat{X}_3 = b_n\partial_{u_n} \\ &\widehat{X}_4 = b_n t\partial_{u_n}, \qquad \widehat{X}_5 = \partial_t, \qquad \widehat{X}_6 = t\partial_t + \tfrac{1}{2}u_n\partial_{u_n}, \qquad \widehat{X}_7 = t^2\partial_t + tu_n\partial_{u_n} \end{aligned} \tag{2.64}$$

with $b_{n,t} = 0$, $b_{n+1} \neq b_n$. The corresponding invariant nonlinear differential equation is:

$$u_{n,tt} = \frac{\gamma_n}{[(b_{n+1} - b_n)u_{n-1} + (b_{n-1} - b_{n+1})u_n + (b_n - b_{n-1})u_{n+1}]^3} \tag{2.65}$$

where γ_n and b_n are arbitrary n-dependent constants.

In [268], the integrability conditions for equations belonging to the class (2.50) have been considered. It has been shown that any equation of this class which has local generalized symmetries can be reduced by point transformations of the form

$$\tilde{u}_n = \sigma_n(t, u_n), \qquad \tilde{t} = \theta(t) \tag{2.66}$$

to either the Toda equation (2.3) or to the potential Toda equation

$$u_{n,tt} = \mathrm{e}^{u_{n+1} - 2u_n + u_{n-1}}. \tag{2.67}$$

2.5. *Symmetries of the two-dimensional Toda equation*

Let us now apply the techniques introduced in section 2.2 to the two-dimensional Toda system (TDTS)

$$\Delta_{\mathrm{TDTS}} = u_{n,xt} - \mathrm{e}^{u_{n-1} - u_n} + \mathrm{e}^{u_n - u_{n+1}} = 0 \tag{2.68}$$

where $u_n = u_n(x, t)$. We also impose that the lattice be invariant, i.e. x and t are independent of n and $\tilde{n} = n$. The TDTS was proposed and studied by Mikhailov [195] and Fordy and Gibbons [79] and is an integrable DΔE, having a Lax pair, infinitely many conservation laws, Bäcklund transformations, soliton solutions and all the usual attributes of integrability [1, 6, 35].

The continuous symmetries for equation (2.68) are obtained by considering the infinitesimal symmetry generator

$$\hat{X} = \xi_n(x, t, u_n)\partial_x + \tau_n(x, t, u_n)\partial_t + \phi_n(x, t, u_n)\partial_{u_n}. \tag{2.69}$$

From the invariance condition (2.17) we get

$$\tau_n = f(t), \qquad \xi_n = h(x), \qquad \phi_n = (h_{,x} + f_{,t})n + g(t) + k(x), \tag{2.70}$$

where $f(t)$, $g(t)$, $h(x)$ and $k(x)$ are arbitrary C^∞ functions. A basis for the symmetry algebra is given by

$$\begin{aligned} T(f) &= f(t)\partial_t + nf_{,t}\partial_{u_n}, \qquad & U(g) &= g(t)\partial_{u_n}, \\ X(h) &= h(x)\partial_x + nh_{,x}\partial_{u_n}, \qquad & W(k) &= k(x)\partial_{u_n}, \end{aligned} \tag{2.71}$$

where, to avoid redundancy, we must impose $k_{,x} \neq 0$.

The nonzero commutation relations are

$$\begin{aligned} &[T(f_1), T(f_2)] = T(f_1 f_{2,t} - f_{1,t} f_2), \qquad [T(f), U(g)] = U(f g_{,t}), \\ &[X(h_1), X(h_2)] = X(h_1 h_{2,x} - h_{1,x} h_2), \\ &[X(h), W(k)] = \begin{cases} W(hk_{,x}), & (hk_{,x})_{,x} \neq 0 \\ cU(1), & (hk_{,x})_{,x} = 0, \qquad hk_{,x} = c. \end{cases} \end{aligned} \tag{2.72}$$

Thus $\{T(f), U(g)\}$ form a Kac–Moody–Virasoro $\hat{u}(1)$ algebra, as do $\{X(h), W(k), h_{,x} \neq 0, U(1)\}$. However the two $\hat{u}(1)$ algebras are not disjoint. This Kac–Moody–Virasoro character of the symmetry algebra is found also in the case of a (2 + 1)-dimensional Volterra equation [237]. It is also characteristic of many other integrable equations involving three continuous variables, such as the Kadomtsev–Petviashvili or three-wave equations [43, 54, 55, 163, 194, 217, 261]. From the symmetry algebra we can construct the group of symmetry transformations which leave the TDTS (2.68) invariant and transform noninvariant solutions into new solutions. Moreover, we can use the subgroups to reduce the TDTS (2.68) to equations in a lower dimensional space.

By adding to (2.68) the equation

$$\Delta_n^{(1)} = \xi(x, t, u_n)u_{n,x} + \tau(x, t, u_n)u_{n,t} - \phi_n(x, t, u_n) = 0 \tag{2.73}$$

we obtain the conditional symmetries. It is easy to show that conditional symmetries in the intrinsic method do not provide any further symmetry reduction. It is, however, worthwhile noting that conditional symmetries in the differential equation method, when

$$\hat{X} = \xi(x, t, u_j(x, t))\partial_x + \tau(x, t, u_j(x, t))\partial_t + \sum_k \phi_k(x, t, u_j(x, t))\partial_{u_k}, \tag{2.74}$$

contain the Bäcklund transformation of the TDTS. In fact, by choosing $\xi = 0, \tau = 1$, equation (2.73) reduces to $\Delta_n^{(1)} = u_{n,t} - \phi_n(x, t, u_j) = 0$ and the determining equations are solved by putting

$$\phi_n(x, t, u_j) = f_{n,t}(x, t) + a[\mathrm{e}^{u_n - f_{n+1}} - \mathrm{e}^{u_{n-1} - f_n}] \tag{2.75}$$

where a is a real constant and $f_n(x, t) = \tilde{u}_n$ is a set of functions which solves the TDTS (2.68). The Bäcklund transformation for the TDTS can be indeed written as [79]

$$u_{n,t} - \tilde{u}_{n,t} = a[\mathrm{e}^{u_n - \tilde{u}_{n+1}} - \mathrm{e}^{u_{n-1} - \tilde{u}_n}], \qquad u_{n-1,x} - \tilde{u}_{n,x} = \frac{1}{a}[\mathrm{e}^{\tilde{u}_{n-1} - u_{n-1}} - \mathrm{e}^{\tilde{u}_n - u_n}]. \tag{2.76}$$

3. Generalized point symmetries on a fixed uniform lattice

We saw in section 2 that point symmetries for purely difference equations on a fixed uniform lattice do not provide very powerful tools, though they work well for differential–difference equations.

Here we shall consider an approach that makes use of a certain type of generalized symmetries that act simultaneously at more than one point of the lattice. We call them 'generalized point symmetries', because in the continuous limit they reduce to point symmetries.

This approach is directly applicable to linear difference equations, indirectly to nonlinear equations that are linearizable by a change of variables.

The underlying formalism is called 'umbral calculus', or 'finite operator calculus'. Its modern development is mainly due to Rota and his collaborators [232–234]. For a review article with an extensive list of references, see [56]. Umbral calculus has been implicitly used in mathematical physics [75, 76, 94, 145, 147, 162]. The only explicit use in physics that we know of is in [44, 57, 156, 157].

3.1. Basic concepts of umbral calculus

Definition 1. *A shift operator T_σ is a linear operator acting on polynomials or formal power series $f(x)$ in the following manner:*

$$T_\sigma f(x) = f(x+\sigma), \qquad x \in \mathbb{R}, \quad \sigma \in \mathbb{R}. \tag{3.1}$$

For functions of several variables we introduce shift operators in the same manner

$$T_{\sigma_i} f(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n) = f(x_1, \ldots, x_{i-1}, x_i + \sigma_i, x_{i+1}, \ldots, x_n). \tag{3.2}$$

In this section, unless explicitly stated, we restrict the exposition to the case of one real variable $x \in \mathbb{R}$. The extension to n variables and other fields is obvious. We will sometimes drop the subscript on the shift operator T when that does not give rise to misinterpretations.

Definition 2. *An operator U is called a delta operator if it satisfies the following properties,*

(1) It is shift invariant;

$$T_\sigma U = U T_\sigma, \qquad \forall \sigma \in \mathbb{R}, \tag{3.3}$$

(2)

$$Ux = c \neq 0, \qquad c = const, \tag{3.4}$$

(3)

$$Ua = 0, \qquad a = const, \tag{3.5}$$

and the kernel of U consists precisely of constants.

Important properties of delta operators are:

1. For every delta operator U there exists a unique series of basic polynomials $\{P_n(x)\}$ satisfying

$$P_0(x) = 1 \quad P_n(0) = 0, \qquad n \geqslant 1, \qquad U P_n(x) = n P_{n-1}(x). \tag{3.6}$$

2. For every delta operator U there exists a conjugate operator β, such that

$$[U, x\beta] = 1. \tag{3.7}$$

The operator β satisfies

$$\beta = (U')^{-1}, \qquad U' = [U, x]. \tag{3.8}$$

Equation (3.7) can be interpreted as the Heisenberg relation between the delta operator U and its conjugate $x\beta$.

We shall make use of two types of delta operators. The first is the ordinary (continuous) derivative operator, for which we have

$$U = \partial_x, \qquad \beta = 1, \qquad P_n(x) = x^n. \tag{3.9}$$

The second is a general *difference operator* which we define as

$$U = \Delta = \frac{1}{\sigma}\sum_{k=l}^{m} a_k T_\sigma^k, \qquad l, m \in \mathbb{Z}, \quad l < m, \tag{3.10}$$

where a_k and σ are real constants and T_σ is a shift operator. In order for Δ to be a delta operator, we must impose

$$\sum_{k=l}^{m} a_k = 0. \tag{3.11}$$

We shall also require that the continuous limit of Δ be ∂_x. This imposes a further condition, namely

$$\sum_{k=l}^{m} k a_k = 1. \tag{3.12}$$

Than equations (3.3), . . . , (3.5) hold, with $c = 1$.

If equations (3.11) and (3.12) are satisfied, we shall call Δ a *difference operator of order* $m - l$. Acting with Δ on an arbitrary smooth function $f(x)$ we have

$$\begin{aligned} \Delta f(x) &= \frac{1}{\sigma}\sum_{k=l}^{m} a_k f(x + k\sigma) = \frac{1}{\sigma}\sum_{n=0}^{\infty} \frac{f^{(n)}(x)}{n!}\sigma^n \gamma_n, \\ \gamma_n &= \sum_{k=l}^{m} a_k k^n, \qquad \gamma_0 = 0, \quad \gamma_1 = 1, \end{aligned} \tag{3.13}$$

where $f^{(n)}(x) = \frac{\mathrm{d}^n f(x)}{\mathrm{d}x^n}$. Thus we have

$$\Delta f(x) = \frac{\mathrm{d}f}{\mathrm{d}x} + \sum_{n=2}^{\infty} \frac{f^{(n)}(x)}{n!}\sigma^{n-1}\gamma_n. \tag{3.14}$$

For $m - l \geqslant 2$ we can impose further conditions, namely

$$\gamma_n = 0, \qquad n = 2, 3, \ldots, m - l, \tag{3.15}$$

and obtain

$$\Delta = \frac{\mathrm{d}}{\mathrm{d}x} + O(\sigma^{m-l}). \tag{3.16}$$

From (3.8) the operator β conjugate to Δ is

$$\beta = \left(\sum_{k=l}^{m} a_k k T_\sigma^k\right)^{-1} \tag{3.17}$$

and the basic polynomials are

$$P_n(x) = (x\beta)^n \cdot 1. \tag{3.18}$$

It was shown in [156] that equation (3.18) yields a well-defined polynomial of order n in x. For any differential operator Δ all coefficients in $P_n(x)$ are finite and each involves a finite number of positive powers of shifts in σ.

The simplest examples of difference operators and the related quantities are

1.
$$\begin{aligned} &\Delta^+ = \frac{T-1}{\sigma}, \qquad \beta = T^{-1}, \\ &P_n(x) = (x)_n = x[x-\sigma][x-2\sigma]\ldots[x-(n-1)\sigma], \qquad n \geqslant 1 \end{aligned} \tag{3.19}$$
Order $m - l = 1$.

2.
$$\begin{aligned} &\Delta^- = \frac{1-T^{-1}}{\sigma}, \qquad \beta = T, \\ &P_n(x) = (x)^n = x[x+\sigma][x+2\sigma]\ldots[x+(n-1)\sigma], \qquad n \geqslant 1 \end{aligned} \tag{3.20}$$
Order $m - l = 1$.

3.
$$\begin{aligned} &\Delta^s = \frac{T-T^{-1}}{2\sigma}, \qquad \beta = \left(\frac{T+T^{-1}}{2}\right)^{-1}, \\ &P_{2n}(x) = x^2[x^2-4\sigma^2]\ldots[x^2-(2n-2)^2\sigma^2], \qquad n \geqslant 1, \\ &P_{2n+1}(x) = x[x^2-\sigma^2]\ldots[x^2-(2n-1)^2\sigma^2], \qquad n \geqslant 1. \end{aligned} \tag{3.21}$$
Order $m - l = 2$.

For any Δ we have $P_0 = 1$ and $P_1 = x$.

An important tool in the umbral calculus is the *umbral correspondence*. This is a bijective mapping M between two delta operators U_1 and U_2. This will induce a mapping between the corresponding operators β_1 and β_2, and also between the corresponding basic polynomials:

$$U_1 \overset{M}{\longleftrightarrow} U_2, \qquad \beta_1 \overset{M}{\longleftrightarrow} \beta_2, \qquad P_n^{(1)} \overset{M}{\longleftrightarrow} P_n^{(2)}. \tag{3.22}$$

Let us now consider linear operators $L(x\beta, U)$ that are polynomials, or formal power series in $x\beta$ and U. Since the umbral correspondence preserves the Heisenberg commutation relation (3.7), it will also preserve commutation relations between the operators $L(x\beta, U)$. In particular, we can take

$$U_1 = \partial_x, \qquad \beta_1 = 1, \qquad U_2 = \Delta, \qquad \beta_2 = \beta, \tag{3.23}$$

where Δ is any one of the difference operators (3.10) and β is as in (3.17).

Let A_1 be a Lie algebra of vector fields of the form

$$\widehat{X}_\alpha = \sum_{j=1}^{n} a_\alpha^j(x_1, \ldots, x_n)\partial_{x_j}, \qquad [\widehat{X}_\alpha, \widehat{X}_\beta] = c_{\alpha\beta}^\gamma \widehat{X}_\gamma. \tag{3.24}$$

The umbral correspondence will map this algebra onto an isomorphic Lie algebra A_2 of difference operators

$$\widehat{X}_\alpha^u = \sum_{j=1}^{n} a_\alpha^j(x_1\beta_1, \ldots, x_n\beta_n)\Delta_{x_j}. \tag{3.25}$$

3.2. Umbral calculus and symmetries of linear difference equations

Lie point symmetries of linear differential equations can be expressed in terms of commuting linear operators. Indeed, let us consider a linear differential equation

$$Lu = 0 \tag{3.26}$$

where L is some linear differential operator. The Lie point symmetry algebra of this equation can be realized by evolutionary vector fields of the form (1.11), satisfying equation (1.13). If (3.26) is an ordinary differential equation of order 3 or higher, or a partial differential equation of order 2 or higher, then the characteristic Q of the vector field (1.11) will have a specific form. Using vector notation $\overrightarrow{x} = (x_1, x_2, \ldots, x_n)$ we can write the characteristic as

$$Q = f(\overrightarrow{x}) + \widehat{X}u, \tag{3.27}$$

where $f(\overrightarrow{x})$ is the general solution of equation (3.26) and $\widehat{X}$ is a first-order linear operator of the form [19]

$$\widehat{X} = \sum_{i=1}^{n} \xi_i(\overrightarrow{x})\partial_{x_i} - \phi(\overrightarrow{x}), \tag{3.28}$$

satisfying

$$[L, \widehat{X}]u|_{Lu=0} = 0. \tag{3.29}$$

Using the umbral correspondence, we can carry this result over to a class of linear difference equations. Indeed, let us associate a difference operator L_D with L by the umbral correspondence $\partial_{x_i} \to \Delta_{x_i}, x_i \to x_i\beta_i$. Equation (3.26) is replaced by a difference equation

$$L_D u = 0. \tag{3.30}$$

The analogue of equation (3.29) will hold, namely

$$[L_D, \widehat{X}_D]u|_{L_D u=0} = 0. \tag{3.31}$$

The difference operator $\widehat{X}_D$ will not generate point transformations taking solutions into solutions. These difference operators do however provide commuting flows, i.e. difference equations compatible with equation (3.30).

3.3. Symmetries of the linear time-dependent Schrödinger equation on a lattice

As an example let us apply the umbral correspondence to the free time-dependent Schrödinger equation

$$\mathrm{i}\frac{\partial\psi}{\partial t} + \sum_{i=1}^{n} \frac{\partial^2\psi}{\partial x_i^2} = 0. \tag{3.32}$$

We obtain the difference equation

$$L_D\psi = 0, \qquad L_D = \mathrm{i}\Delta_t + \frac{1}{2}\sum_{k=1}^{n}(\Delta_{x_k})^2, \tag{3.33}$$

where Δ_t and Δ_{x_k} are any of the difference operators introduced in section 3.1. The symmetries of equation (3.33) will be represented in terms of the 'discrete' evolutionary vector fields

$$\widehat{X}_D^E = Q_D\partial_\psi + Q_D^*\partial_{\psi^*}, \qquad Q_D = \eta_D - \tau_D\Delta_t\psi - \xi_D^k\Delta_{x_k}\psi, \tag{3.34}$$

where η_D, τ_D and ξ_D^k are functions of $t\beta_t, x_j\beta_j$ and ψ, ψ^* (the $*$ indicates complex conjugation).

As in the continuous case, it can be shown [156] that the characteristic Q will in this case have the form

$$Q_D = \chi(x_k\beta_k, t\beta_t) + \mathrm{i}\widehat{X}_D\psi, \tag{3.35}$$

where χ is the general solution of equation (3.33). The first-order difference operator $\widehat{X}_D$ satisfies

$$\widehat{X}_D = \mathrm{i}\left[\tau_D(t\beta_t)\Delta_t + \sum_{k=1}^{n}\xi_D^k\Delta_{x_k} - \mathrm{i}\eta_D\right], \tag{3.36}$$

$$[L_D, \widehat{X}_D]\psi|_{L_D\psi=0} = 0. \tag{3.37}$$

Explicitating and solving equation (3.37), we obtain a difference realization of the Schrödinger algebra, first obtained in the continuous case by Niederer [200]. A basis for this algebra is given by the following operators:

$$\begin{aligned}
&\hat{P}_0 = \Delta_t, \qquad \hat{P}_j = \Delta_{x_j}, \qquad \hat{L}_{j,k} = (x_j\beta_j)\Delta_{x_k} - (x_k\beta_k)\Delta_{x_j},\\
&\hat{B}_k = (t\beta_t)\Delta_{x_k} - \frac{\mathrm{i}}{2}x_k\beta_k, \qquad \hat{D} = 2(t\beta_t)\Delta_t - \sum_{k=1}^{n}(x_k\beta_k)\Delta_{x_k} + \frac{1}{2},\\
&\hat{C} = (t\beta_t)^2\Delta_t + \sum_{k=1}^{n}(t\beta_t)(x_k\beta_k)\Delta_{x_k} + \frac{1}{2}t\beta_t - \frac{\mathrm{i}n}{4}\sum_{k=1}^{n}(x_k\beta_k)^2, \qquad \hat{W} = \mathrm{i}.
\end{aligned} \tag{3.38}$$

Comparing with the continuous limit (or using the umbral correspondence), we see that $\hat{P}_0$, $\hat{P}_j$ correspond to time and space translation, $\hat{L}_{j,k}$ to rotations, $\hat{B}_k$ to Galilei boosts, $\hat{D}$ and $\hat{C}$ to dilations and projective transformations and $\hat{W}$ to changes of phase of the wavefunction.

3.4. Symmetries of the discrete heat equation

As a further example let us consider the discrete heat equation

$$\Delta_{xx}u(x,t) - \Delta_t u(x,t) = 0. \tag{3.39}$$

Equation (3.39) is a linear partial difference equation on a two-dimensional lattice. Floreanini, Negro, Nieto and Vinet showed in [75] when $\Delta = \Delta^+$ that (3.39) has a symmetry algebra isomorphic to that of the continuous heat equation

$$u_{,xx}(x,t) - u_{,t}(x,t) = 0. \tag{3.40}$$

This result can easily be recovered using the umbral calculus [145, 162].

Since (3.39) is linear, the symmetries are obtained by considering an evolutionary vector field of the form

$$\hat{X}_e = Q\partial_u = (\tau\Delta_t + \xi\Delta_x + f)u\partial_u \tag{3.41}$$

and the determining equation is given by

$$\Delta_t Q - \Delta_{xx}Q|_{\Delta_{xx}u=\Delta_t u} = 0 \tag{3.42}$$

whose explicit expression is

$$\Delta_t(\xi\Delta_x u + \tau\Delta_t u + fu) - \Delta_{xx}(\xi\Delta_x u + \tau\Delta_t u + fu)|_{\Delta_{xx}u=\Delta_t u} = 0. \tag{3.43}$$

Defining $Df = [\Delta, f]$ it is immediate to see that $(Df)\cdot 1 = (\Delta f)\cdot 1$ and for the operator D the Leibniz rule is satisfied: $Dfg = (Df)g + f(Dg)$. In terms of the operator D we can split equation (3.43) into the following overdetermined system of equations:

$$\begin{aligned}
&D_x\tau = 0, && D_t\tau - 2D_x\xi = 0\\
&D_t\xi - D_{xx}\xi - 2D_x f = 0, && D_t f - D_{xx}f = 0.
\end{aligned} \tag{3.44}$$

From the umbral correspondence the solution of equation (3.44) is

$$\begin{aligned} \tau &= \tau_2(t\beta_t)^2 + \tau_1 t\beta_t + \tau_0 \\ \xi &= \tfrac{1}{2}(\tau_1 + 2\tau_2 t\beta_t)x\beta_x + \xi_1 t\beta_t + \xi_0 \\ f &= \tau_2\left[\tfrac{1}{4}(x\beta_x)^2 + \tfrac{1}{2}t\beta_t\right] + \tfrac{1}{2}\xi_1 x\beta_x + f_0 \end{aligned} \tag{3.45}$$

where $\tau_0, \tau_1, \tau_2, \xi_1, \xi_0$ and f_0 are arbitrary constants, functions of the lattice spacing and shifts. By a suitable choice of these constants, we get the following representation of the symmetries:

$$\begin{aligned} &\hat{P}_0 = (\Delta_t u)\partial_u, \qquad \hat{P}_1 = (\Delta_x u)\partial_u \\ &\hat{W} = u\partial_u, \qquad \hat{B} = [2t\beta_t\Delta_x u + x\beta_x u]\partial_u \\ &\hat{D} = \left[2t\beta_t\Delta_t u + x\beta_x\Delta_x u + \tfrac{1}{2}u\right]\partial_u \\ &\hat{K} = \left[(t\beta_t)^2\Delta_t u + t\beta_t x\beta_x\Delta_x u + \left(\tfrac{1}{4}(x\beta_x)^2 + \tfrac{1}{2}t\beta_t\right)u\right]\partial_u. \end{aligned} \tag{3.46}$$

The results obtained above show the power and generality of umbral calculus. Starting from the symmetry algebra of the continuous heat equation (known already to Sophus Lie) and applying the umbral correspondence, we have obtained the symmetry algebra of the discrete heat equation for a very large class of discretizations. Indeed, each Δ is any one of the difference operators (3.10) of section 3.1 and β is the corresponding conjugate operator. Without the umbral calculus each discretization must be considered separately [75, 76, 162] and it is not at all obvious that all discretizations have the same symmetry algebra as the continuous equation.

3.5. Discretization of a relativistic wave equation

Umbral calculus not only allows us to discretize linear equations while preserving their symmetry algebras, but also allows us to solve the obtained difference equations. To see how this works and how linear techniques, such as the separation of variables, can be adapted to difference equations, let us consider a relativistic wave equation in two dimensions [156].

In the continuous case we write the equation in light cone variables as

$$\hat{R}\phi(x,y) = \frac{\partial^2}{\partial_x\partial_y}\phi(x,y) = -k\phi(x,y). \tag{3.47}$$

Its symmetry algebra has basis

$$\hat{P}_1 = \partial_x, \qquad \hat{P}_2 = \partial_y, \qquad \hat{M} = x\partial_x - y\partial_y, \tag{3.48}$$

corresponding to translations and Lorentz transformations.

Since the wave operator $\hat{R}$ and the operator $\hat{M}$, generating Lorentz transformations, commute, we can construct a complete set of common eigenfunctions, satisfying (3.47) and

$$\hat{M}\phi = \lambda\phi. \tag{3.49}$$

The natural approach to this system would be to introduce a hyperbolic version of the polar coordinates, namely

$$x = \frac{\rho}{2}\,\mathrm{e}^{\alpha}, \qquad y = \frac{\rho}{2}\,\mathrm{e}^{-\alpha}, \tag{3.50}$$

where α is an ignorable variable. Instead, having in mind the umbral correspondence, we look for a series solution of equation (3.47) in terms of solutions of equation (3.49):

$$\phi_\lambda(x,y) = \sum_{n=0}^{\infty} a_n x^{n+\lambda} y^n. \tag{3.51}$$

Substituting into (3.47) we determine a_n and obtain

$$\phi_{k\lambda}(x, y) = x^\lambda \sum_{n=0}^{\infty} (-k)^n \frac{1}{\Gamma(\lambda + n + 1)n!} (xy)^n, \tag{3.52}$$

or in the coordinates (3.50):

$$\phi_{k\lambda}(\rho) = \mathrm{e}^{\lambda\alpha} k^{-\frac{\lambda}{2}} J_\lambda(\sqrt{k\rho}), \tag{3.53}$$

where J_λ is a Bessel function.

The umbral version of equations (3.47) and (3.49) is

$$\Delta_x \Delta_y \psi = -k\psi, \qquad (x\beta_x \Delta_x - y\beta_y \Delta_y)\psi = \lambda\psi, \tag{3.54}$$

where again Δ_x, Δ_y are any difference operators with respect to the corresponding variable. Using the umbral correspondence and equation (3.52) we immediately obtain a formal solution of the system (3.54), namely

$$\psi_{k\lambda}(x, y) = (x\beta_x)^\lambda \sum_{n=0}^{\infty} (-k)^n \frac{1}{\Gamma(\lambda + n + 1)n!} (x\beta_x)^n (y\beta_y)^n \cdot 1. \tag{3.55}$$

Here $\psi_{k\lambda}(x, y)$ is a function, since all the operators β appearing on the right-hand side of equation (3.55) are applied to a constant and we have e.g. $\beta_x \cdot 1 = 1$.

Two comments are in order here.

1. The series (3.52) converges for any value of x and y. In the discrete case the series (3.55) is a formal solution for any choice of the difference operator Δ. However, the convergence properties of the series depend on the choice of Δ. For instance, if Δ is Δ^+ or Δ^- of equation (3.19), or (3.20), then we have $\beta^+ = T^{-1}$, or $\beta^- = T$ respectively. The series (3.55) will converge in both cases for all values of x and y; however, the lattice spacing must satisfy

$$\sigma_x \sigma_y < \frac{1}{k}. \tag{3.56}$$

2. The expansion (3.55), contrary to (3.52), does not correspond directly to the separation of variables in the coordinate (3.50) since the expression $[(x\beta_x)(y\beta_y)]^n$ is not a function of xy alone. For instance for $n = 2$ and $\beta_x = T_x^{-1}$, $\beta_y = T_y^{-1}$ we have

$$\left(xT_x^{-1}\right)^2 \left(yT_y^{-1}\right)^2 = x(x - \sigma_x)y(y - \sigma_y)T_x^{-2}T_y^{-2}.$$

4. Generalized symmetries on fixed uniform lattices

4.1. Generalized symmetries of difference equations

We consider here the construction of generalized symmetries for integrable difference equations and show the structure of the infinite-dimensional Lie algebra of point and higher symmetries. We do so on an explicit example of a nonlinear difference equation known to be integrable to provide a clear idea of the structure of the symmetry algebra. In particular we will show that, for equations defined on a lattice, a subset of the generators of the infinite symmetry algebra contracts in the continuum limit to the Lie algebra of a Lie point symmetry group. We prove that in this approach the scaling symmetries are generalized symmetries.

We will not present the entire machinery of integrable systems here, nor the most general definition of an integrable system. Good reviews on integrable discrete equations and on their physical and numerical applications can be found in the literature [36, 46, 78, 250, 252–254]. We will just consider here the minimal amount of ideas and results necessary for readers who are not experts on soliton theory to understand the results on generalized symmetries of discrete integrable systems presented below.

Generalized symmetries of difference equations are symmetries whose infinitesimal generators depend on more than one point of the lattice and on derivatives of the continuous variables. Noether was the first to notice in 1918 [208] that one can extend symmetries of a differential equation by including higher derivatives of the dependent variables in the transformation. They are more rare than point symmetries. We can obtain an infinite number of generalized symmetries [211] when the system is integrable [1, 3, 6, 35, 71, 83–86, 199, 209, 269], i.e. when the equations can be written as the compatibility condition for an overdetermined system of linear equations (the Lax pair) and can be linearized either directly or via an inverse scattering transform.

Let us first review the situation in the case of the Korteweg–de Vries equation

$$u_t(x,t) = u_{xxx}(x,t) - 6u(x,t)u_x(x,t), \tag{4.1}$$

the prototype of the integrable partial differential equations. Equation (4.1) is characterized by a nontrivial Lax pair [138]

$$L(u)\psi = \lambda\psi, \tag{4.2}$$

$$\psi_t = -M(u)\psi, \tag{4.3}$$

a system of linear equations compatible only on the solution set of equation (4.1). In equations (4.2) and (4.3) λ is an eigenvalue, L and M are two linear operators with coefficients depending on u but not on λ

$$L(u) = -\partial_{xx} + u(x,t), \qquad \lambda = k^2, \tag{4.4}$$

$$\lambda_t = 0, \qquad M(u) = -4\partial_{xxx} + 6u(x,t)\partial_x + 3u_x(x,t). \tag{4.5}$$

The function ψ, often called the wave or spectral function, depends on the independent variables (x,t) and on λ. If $\lambda_t = 0$ then the compatible system is said to be *isospectral* and λ is an integral of motion, together with all the functions which depend only on it. The compatibility of equations (4.2) and (4.3) implies the Lax equation

$$L_t(u) = [L(u), M(u)] \tag{4.6}$$

if $\lambda_t = 0$ or

$$L_t(u) = [L(u), M(u)] + f(L(u), t) \tag{4.7}$$

for $\lambda_t = f(\lambda, t)$. Here $f(z,t)$ is an entire function of its first argument.

In the particular case when equation (4.2) is the Schrödinger spectral problem (4.4) and $u(x)$ vanishes at infinity, the solution of equation (4.4), for $k \in \mathcal{R}$, has the following asymptotic behaviour:

$$\psi(x,k) \to \mathrm{e}^{-\mathrm{i}kx} + R(k)\,\mathrm{e}^{\mathrm{i}kx}, \qquad (x \to +\infty), \tag{4.8}$$

$$\psi(x,k) \to T(k)\,\mathrm{e}^{-\mathrm{i}kx}, \qquad (x \to -\infty), \tag{4.9}$$

where $T(k)$ is the transmission and $R(k)$ is the reflection coefficient. It can easily be proven that, when $u(x,t)$ evolves according to the KdV equation (4.1) [35] the function $T(k,t)$ is conserved, i.e. $\dot{T}(k,t) = 0$ and the reflection coefficient evolves according to the equation

$$\dot{R}(k,t) = -8\,\mathrm{i}k^3 R(k,t). \tag{4.10}$$

For $k_j = \mathrm{i}p_j$, $j = 1, 2, \ldots, N$, the function $\psi(x, p_j) = f_j(x)$ is bounded at infinity and we can define its normalization coefficient, ρ_j, as

$$\rho_j = \left[\int_{-\infty}^{\infty} \mathrm{d}x f_j(x)^2\right]^{-1} \qquad j = 1, 2, \ldots, N. \tag{4.11}$$

The spectral transform $\mathcal{S}$ of the function $u(x)$ is, by definition, the collection of data

$$\mathcal{S}[u] = \{R(k, t), -\infty < k < \infty; p_j, \rho_j(t), j = 1, 2, \ldots, N\}. \tag{4.12}$$

If we can associate a denumerable set of operators $M_m(u)$, $m = 1, 2, \ldots$, with the operator $L(u)$ then we will obtain from the Lax equations (4.6), (4.7) a denumerable set of equations, i.e. *a hierarchy of equations*. In the case of the Schrödinger spectral problem (4.4) the hierarchy of equations is written down in terms of a recursion operator $\mathcal{L}$, obtained in an algorithmic way from (4.4) [28, 35]

$$\mathcal{L}\psi(x) = \psi_{xx}(x) - 4u(x, t)\psi(x) + 2u_x(x, t)\int_x^{\infty} \mathrm{d}y\psi(y), \tag{4.13}$$

and reads:

$$u_t(x, t) = \alpha(\mathcal{L}, t)u_x(x, t) + \beta(\mathcal{L}, t)[xu_x(x, t) + 2u(x, t)]. \tag{4.14}$$

The entire (with respect to the first argument) functions α and β characterize the equation of the hierarchy. If only α is present then $\dot{\lambda} = 0$. If also β is present then $\dot{\lambda} \neq 0$, and we have

$$\dot{k} = \beta(-4k^2, t)k. \tag{4.15}$$

With equation (4.14) we can associate an evolution of the reflection coefficient $R(k)$

$$\frac{\mathrm{d}R(k, t)}{\mathrm{d}t} = 2\mathrm{i}k\alpha(-4k^2, t)R(k, t), \tag{4.16}$$

where by the symbol $\frac{\mathrm{d}}{\mathrm{d}t}$ we mean the total derivative with respect to t.

Lie symmetries, both point and generalized [211], can be written as flows commuting with the equation under study, i.e. equations of the form

$$u_\epsilon(x, t) = F(u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}, \ldots), \qquad u = u(x, t) = u(x, t; \epsilon), \tag{4.17}$$

such that $u_{\epsilon t}(x, t; \epsilon) = u_{t\epsilon}(x, t; \epsilon)$. In equation (4.17) ϵ plays the role of the group parameter. As the KdV hierarchy (4.14) is given by evolutionary equations, in all generality its symmetries can be written as

$$u_\epsilon(x, t; \epsilon) = F_l(u, u_x, u_{xx}, u_{xxx}, \ldots, u_{lx}). \tag{4.18}$$

Equation (4.18) is an evolutionary equation in the 'time' ϵ and thus any equation of the hierarchy (4.14) is a symmetry for the KdV if it commutes with it. By taking into account the one-to-one correspondence between the potential u and the spectrum $\mathcal{S}[u]$ one can easily prove, by checking the commutativity of the spectrum, that the isospectral equations

$$u_{\epsilon_j}(x, t; \epsilon) = \mathcal{L}^j u_x(x, t; \epsilon) \tag{4.19}$$

are symmetries. If $\beta \neq 0$ then $\dot{\lambda} \neq 0$ and only when

$$\alpha(-4k^2, t) = \alpha_0(-4k^2) - 24k^2\beta(-4k^2)t \tag{4.20}$$

the corresponding non-isospectral equation

$$u_\epsilon(x, t; \epsilon) = \alpha(\mathcal{L}, t)u_x(x, t) + \beta(\mathcal{L})[xu_x(x, t) + 2u(x, t)] \tag{4.21}$$

is a symmetry. In particular, choosing $\beta(\mathcal{L}) = 1$ and $\alpha_0 = 0$ from equation (4.20) we get $\alpha(\mathcal{L}, t) = 6t\mathcal{L}$ and the symmetry reads:

$$u_\epsilon(x, t; \epsilon) = 6tu_t(x, t) + xu_x(x, t) + 2u(x, t), \tag{4.22}$$

the usual dilation symmetry of the KdV equation [211]. The higher order non-isospectral symmetries, when $\frac{\mathrm{d}\beta(\mathcal{L})}{\mathrm{d}\mathcal{L}} \neq 0$, will all be nonlocal due to the form of the recursion operator (4.13).

We have described here the integrability procedure in the case of partial differential equations, where it was first introduced. This procedure has been extended to the case of differential–difference and difference–difference equations [4, 5, 10, 45, 58, 72, 73, 183, 192, 202, 205, 244, 245].

In the case of an integrable differential–difference equation

$$u_{n,t}(t) = E(n, t, u_n(t), u_{n+1}(t), \ldots, u_{n+k}(t)) \tag{4.23}$$

the linear operators L and M that characterize it, depend on the shift operators in the discrete variable n. The Lax equations (4.6) and (4.7) are still valid. The recursion operator $\mathcal{L}$ and the Lax pair will depend on the shift operators, instead of x derivatives. This implies that the higher equations of the hierarchy

$$u_{n,t}(t) = E_j(n, t, u_n(t), u_{n+1}(t), \ldots, u_{n+k_j}(t)) \tag{4.24}$$

and the higher symmetries will depend on points further away from the point n instead of depending on higher derivatives. The situation is slightly different in the case of difference–difference equations

$$E(n, m, u_{n,m}, u_{n+1,m}, u_{n,m+1}, \ldots) = 0. \tag{4.25}$$

The Lax pair in this case involves two linear operators $L_{n,m}$ and $M_{n,m}$ of the shift operator in n with coefficients depending on $u_{n+j,m+i}$. The linear equation (4.3) is replaced by

$$\psi_{n,m+1} = -M_{n,m}(u)\psi_{n,m}. \tag{4.26}$$

The Lax equation in the isospectral regime ($\lambda_{m+1} = \lambda_m$) now reads:

$$L_{n,m+1}M_{n,m} = M_{n,m}L_{n,m}. \tag{4.27}$$

In the nonisospectral case, when $\lambda_{m+1} = f(\lambda_m)$, with $f(.)$ an entire function of its argument,we have

$$L_{n,m+1}M_{n,m} = M_{n,m}f(L_{n,m}). \tag{4.28}$$

Few results are known on generalized symmetries of difference–difference equations [144, 189, 201]. The point and generalized symmetries for the discrete time Toda Lattice (2.1) were computed in [144] using the techniques presented above. The Lie symmetries are provided by evolutionary equations commuting with the original equation. So, for example, the symmetries of the discrete time Toda Lattice (2.1) are given by the Toda Lattice hierarchy of nonlinear differential–difference equations (4.33), with the evolution not in the time variable, but in the group parameter.

In the following, we will construct an infinite class of symmetries for the Toda lattice. We will construct formally all the generalized symmetries and present explicitly the simplest examples. We will show the structure of the algebra of the generalized and point symmetries, using the one-to-one correspondence between the configuration space and the spectral space, where all equations are linear. In the continuous limit the Lie algebra of the point symmetries of the potential KdV (4.83) is recovered, as are the generalized symmetries. Finally, we present the Bäcklund transformations for the Toda lattice and show their relation to the symmetries [134]. For the corresponding results for the Volterra equation, the completely discrete Toda lattice and for the discrete nonlinear Schrödinger equation we refer to the literature [45, 101, 102, 107, 144, 149].

4.2. The Toda system and its symmetries

The Toda equation (2.3) can be rewritten in the form of a system

$$\dot{a}_n(t) = a_n(t)(b_n(t) - b_{n+1}(t)), \qquad \dot{b}_n(t) = a_{n-1}(t) - a_n(t), \tag{4.29}$$

where

$$b_n = \dot{v}_n, \qquad a_n = \mathrm{e}^{\mathrm{v_n - v_{n+1}}}. \tag{4.30}$$

It can be associated with the discrete Schrödinger spectral problem [27, 38–40, 73]

$$\psi(n-1, t; \lambda) + b_n \psi(n, t; \lambda) + a_n \psi(n+1, t; \lambda) = \lambda \psi(n, t; \lambda). \tag{4.31}$$

The time evolution of the wavefunction $\psi(n, t; \lambda)$ is given by

$$\psi_t(n, t; \lambda) = -a_n(t)\psi(n+1, t; \lambda). \tag{4.32}$$

For the point symmetries of the Toda equation see section 2.3.

With the spectral problem (4.31) we can associate a set of nonlinear differential–difference equations (the Toda lattice hierarchy)

$$\begin{pmatrix} \dot{a}_n \\ \dot{b}_n \end{pmatrix} = f_1(\mathcal{L}, t) \begin{pmatrix} a_n(b_n - b_{n+1}) \\ a_{n-1} - a_n \end{pmatrix}, \tag{4.33}$$

where $f_1(\mathcal{L}, t)$ is an entire function of its first argument and the recursion operator $\mathcal{L}$ is given by

$$\mathcal{L} \begin{pmatrix} p_n \\ q_n \end{pmatrix} = \begin{pmatrix} p_n b_{n+1} + a_n(q_n + q_{n+1}) + (b_n - b_{n+1})s_n \\ b_n q_n + p_n + s_{n-1} - s_n \end{pmatrix}. \tag{4.34}$$

The operator (4.34) was first obtained by Dodd [58] and by Bruschi, Levi and Ragnisco [26]. In equation (4.34) s_n is a solution of the nonhomogeneous first-order equation

$$s_{n+1} = \frac{a_{n+1}}{a_n}(s_n - p_n). \tag{4.35}$$

For any equation of the hierarchy (4.33) we can write an explicit evolution equation for the function $\psi(n, t; \lambda)$ [27, 29, 31] such that λ does not evolve in time. This is possible if the following boundary conditions

$$\lim_{|n|\to\infty} a_n - 1 = \lim_{|n|\to\infty} b_n = \lim_{|n|\to\infty} s_n = 0, \tag{4.36}$$

are imposed on the fields a_n, b_n and s_n. The boundedness of the solutions of equations (4.36) was not required in the literature [26, 29, 31], but it is necessary to get a hierarchy of nonlinear differential–difference equations with well-defined evolution of the spectra. We can than associate with equation (4.31) a spectrum $S[u]$ [27, 72, 73, 192, 253] defined in the complex plane of the variable z ($\lambda = z + z^{-1}$):

$$S[u] = \{R(z, t), z \in \mathbf{C}_1; z_j, c_j(t), |z_j| < 1, j = 1, 2, \ldots, N\}, \tag{4.37}$$

where $R(z, t)$ is the reflection coefficient, $\mathbf{C}_1$ is the unit circle in the complex z plane, z_j are isolated points inside the unit disc and c_j are some complex functions of t related to the residues of $R(z, t)$ at the poles z_j. When a_n, b_n and s_n satisfy the boundary conditions (4.36), the spectral data define the potentials in a unique way. Thus, there is a one-to-one correspondence between the evolution of the potentials (a_n, b_n) of the discrete Schrödinger spectral problem (4.31), given by equation (4.33) and that of the reflection coefficient $R(z, t)$, given by

$$\frac{\mathrm{d}R(z, t)}{\mathrm{d}t} = \mu f_1(\lambda, t) R(z, t), \qquad \mu = z^{-1} - z. \tag{4.38}$$

In equation (4.38) and below, $\frac{\mathrm{d}}{\mathrm{d}t}$ denotes the total derivative with respect to t.

The Toda system is obtained from equation (4.33) by choosing $f_1(\lambda, t) = 1$ and thus the evolution equation of the reflection coefficient is given by

$$\frac{\mathrm{d}R(z,t)}{\mathrm{d}t} = \mu R(z,t). \tag{4.39}$$

The symmetries for the Toda system (4.29), or the Toda equation (2.3) are provided by all flows commuting with the equation itself. Let us introduce the following denumerable set of equations:

$$\begin{pmatrix} a_{n,\epsilon_k} \\ b_{n,\epsilon_k} \end{pmatrix} = \mathcal{L}^k \begin{pmatrix} a_n(b_n - b_{n+1}) \\ a_{n-1} - a_n \end{pmatrix}. \tag{4.40}$$

Here k is any positive integer and ϵ_k is a variable. We can associate with the equation (4.40) an evolution of the reflection coefficient

$$\frac{\mathrm{d}R}{\mathrm{d}\epsilon_k} = \mu\lambda^k R. \tag{4.41}$$

It can be verified directly that the flows (4.39) and (4.41) commute and hence the same must be true for the corresponding equations, i.e. for the Toda system (4.29) and the equations (4.40). This implies that equations (4.40) are symmetries of the Toda system and consequently ϵ_k is a group parameter. From the point of view of the spectral problem (4.31), equations (4.40) correspond to isospectral deformations, i.e. we have $\lambda_{\epsilon_k} = 0$. For any ϵ_k, the solution of the Cauchy problem for equation (4.40) provides a solution of the Toda system (4.29) $(a_n(t, \epsilon_k), b_n(t, \epsilon_k))$ in terms of the initial condition $[a_n(t, \epsilon_k = 0), b_n(t, \epsilon_k = 0)]$. The group transformation corresponding to the group parameter ϵ_k can usually be written explicitly only for the lowest values of k. In the case of the generalized symmetries, the group action is obtained in principle by solving the Cauchy problem for the nonlinear characteristic equation starting from a generic solution of the Toda lattice. This often cannot be done. In all cases we can construct just a few classes of explicit group transformations corresponding to very specific solutions of the Toda lattice equation, namely the solitons, the rational solutions and the periodic solutions [253]. In all cases one can use the symmetries (4.40) to do, for example, symmetry reduction and to reduce the equation under consideration to an ordinary differential equation, or possibly a functional one. This is done by looking for fixed points of the transformation, i.e. putting $a_{n,\epsilon_k} = 0$, $b_{n,\epsilon_k} = 0$,

We can extend the class of symmetries by considering nonisopectral deformations of the spectral problem (4.31) [77, 83, 84, 149, 179]. Thus for the Toda system we obtain

$$\begin{pmatrix} a_{n,\epsilon_k} \\ b_{n,\epsilon_k} \end{pmatrix} = \mathcal{L}^{k+1} t \begin{pmatrix} a_n(b_n - b_{n+1}) \\ a_{n-1} - a_n \end{pmatrix} + \mathcal{L}^k \begin{pmatrix} a_n[(2n+3)b_{n+1} - (2n-1)b_n] \\ b_n^2 - 4 + 2[(n+1)a_n - (n-1)a_{n-1}] \end{pmatrix}. \tag{4.42}$$

In correspondence with equation (4.42) we have the evolution of the reflection coefficient (4.37), given by

$$\frac{\mathrm{d}R}{\mathrm{d}\epsilon_k} = \mu\lambda^{k+1} t R, \qquad \lambda_{\epsilon_k} = \mu^2\lambda^k, \tag{4.43}$$

The proof that equations (4.42) are symmetries is done by showing that the flows (4.43) and (4.39) in the space of the reflection coefficients commute.

In addition to the above two hierarchies of symmetries (4.40) and (4.42), we can construct two further cases, which, however, do not satisfy the asymptotic boundary conditions (4.36). They are:

$$\begin{pmatrix} a_{n,\epsilon} \\ b_{n,\epsilon} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad \begin{pmatrix} a_{n,\epsilon} \\ b_{n,\epsilon} \end{pmatrix} = t \begin{pmatrix} a_n(b_n - b_{n+1}) \\ a_{n-1} - a_n \end{pmatrix} + \begin{pmatrix} 2a_n \\ b_n \end{pmatrix}. \tag{4.44}$$

As these exceptional symmetries do not satisfy the asymptotic boundary conditions (4.36), we cannot write a corresponding evolution equation for the reflection coefficient (4.37).

Let us now write the lowest order symmetries for the Toda system (4.29) that one can get from the hierarchies (4.40), (4.42). In the case of the Toda lattice (2.3) the symmetries are obtained from those of the Toda system (4.29) by using the transformation (4.30). The symmetries of the Toda lattice and the Toda system, corresponding to the isospectral and nonisospectral flows, will have the same evolution of the reflection coefficient. The transformation (4.30) involves an integration (to obtain u_n). The integration constant must be chosen so as to satisfy the following boundary conditions:

$$\lim_{|n|\to\infty} v_n = 0. \tag{4.45}$$

In the case of the exceptional symmetries such an integration will provide an additional symmetry.

Taking $k = 0$, 1 and 2 in equation (4.40) we obtain the first three isospectral symmetries for the Toda system, namely:

$$a_{n,\epsilon_0} = a_n(b_n - b_{n+1}), \qquad b_{n,\epsilon_0} = a_{n-1} - a_n, \tag{4.46}$$

$$a_{n,\epsilon_1} = a_n\big[b_n^2 - b_{n+1}^2 + a_{n-1} - a_{n+1}\big], \qquad b_{n,\epsilon_1} = a_{n-1}[b_n + b_{n-1}] - a_n[b_{n+1} + b_n], \tag{4.47}$$

$$\begin{aligned} a_{n,\epsilon_2} &= a_n\big[b_n^3 - b_{n+1}^3 + a_n b_n - 2a_{n+1}b_{n+1} + a_{n-1}b_{n-1} + 2a_{n-1}b_n \\ &\qquad - a_{n+1}b_{n+2} - a_n b_{n+1} - 2b_n + 2b_{n+1}\big], \\ b_{n,\epsilon_2} &= a_{n-1}\big[b_n^2 + b_{n-1}^2 + b_n b_{n-1} + a_{n-1} + a_{n-2} - 2\big] \\ &\qquad - a_n\big[b_n^2 + b_{n+1}^2 + b_n b_{n+1} + a_{n+1} + a_n - 2\big]. \end{aligned} \tag{4.48}$$

The lowest nonisospectral symmetry is obtained from equation (4.42), taking $k = 0$. It is

$$\begin{aligned} a_{n,\nu} &= a_n\big\{t\big[b_n^2 - b_{n+1}^2 + a_{n-1} - a_{n+1}\big] + (2n+3)b_{n+1} - (2n-1)b_n\big\}, \\ b_{n,\nu} &= t\{a_{n-1}(b_n + b_{n-1}) - a_n(b_{n+1} + b_n)\} + b_n^2 - 4 + 2[(n+1)a_n - (n-1)a_{n-1}]. \end{aligned} \tag{4.49}$$

The nonisospectral symmetries for $k > 0$ are nonlocal.

The exceptional symmetries (4.44) are

$$a_{n,\mu_0} = 0, \qquad b_{n,\mu_0} = 1, \tag{4.50}$$

$$a_{n,\mu_1} = 2a_n + t\dot{a}_n, \qquad b_{n,\mu_1} = b_n + t\dot{b}_n. \tag{4.51}$$

The corresponding symmetries for the Toda lattice are

$$v_{n,\epsilon_0} = \dot{v}_n \tag{4.52}$$

$$v_{n,\epsilon_1} = \dot{v}_n^2 + \mathrm{e}^{v_{n-1}-v_n} + \mathrm{e}^{v_n - v_{n+1}} - 2 \tag{4.53}$$

$$v_{n,\epsilon_2} = \dot{v}_n^3 - 2\dot{v}_n + \mathrm{e}^{v_{n-1}-v_n}(\dot{v}_{n-1} + 2\dot{v}_n) + \mathrm{e}^{v_n - v_{n+1}}(\dot{v}_{n+1} + 2\dot{v}_n) \tag{4.54}$$

$$v_{n,\nu} = t\big\{\dot{v}_n^2 + \mathrm{e}^{v_{n-1}-v_n} + \mathrm{e}^{v_n - v_{n+1}} - 2\big\} - (2n-1)\dot{v}_n + w_n(t), \tag{4.55}$$

where $w_n(t)$ is defined by the following compatible system of equations:

$$w_{n+1}(t) - w_n(t) = -2\dot{v}_{n+1}, \qquad \dot{w}_n(t) = 2(\mathrm{e}^{v_n - v_{n+1}} - 1). \tag{4.56}$$

Under the assumption (4.45) we can integrate equations (4.56) and obtain a formal solution. That is, we can write $w_n(t)$ in the form of an infinite sum;

$$w_n(t) = 2\sum_{j=n+1}^{\infty} \dot{v}_j + \alpha, \tag{4.57}$$

where α is an arbitrary integration constant which can be interpreted as an additional symmetry. The exceptional symmetries read:

$$v_{n,\mu_1} = t\dot{v}_n - 2n \tag{4.58}$$

$$v_{n,\mu_0} = t \tag{4.59}$$

and the additional one, due to the integration

$$v_{n,\mu_{-1}} = 1. \tag{4.60}$$

4.2.1. The symmetry algebra for the Toda lattice. To define the structure of the symmetry algebra for the Toda lattice we need to compute the commutation relations between the symmetries, i.e. between the flows commuting with the equations of the hierarchy. Using the one-to-one correspondence between the integrable equations and the evolution equations for the reflection coefficients, we calculate the commutation relations between the symmetries and thus analyse the structure of the obtained infinite-dimensional Lie algebra.

If we define

$$\mathcal{L}^k = \begin{pmatrix} \mathcal{L}_{11}^{(k)} & \mathcal{L}_{12}^{(k)} \\ \mathcal{L}_{21}^{(k)} & \mathcal{L}_{22}^{(k)} \end{pmatrix}, \tag{4.61}$$

we can write the generators for the isospectral symmetries as

$$\begin{aligned} \hat{X}_k^T = {} & \left\{\mathcal{L}_{11}^{(k)}[a_n(b_n - b_{n+1})] + \mathcal{L}_{12}^{(k)}(a_{n-1} - a_n)\right\}\partial_{a_n} \\ & + \left\{\mathcal{L}_{21}^{(k)}[a_n(b_n - b_{n+1})] + \mathcal{L}_{22}^{(k)}(a_{n-1} - a_n)\right\}\partial_{b_n}. \end{aligned} \tag{4.62}$$

The superscript on the generator $\hat{X}$ is there to indicate that this is the symmetry generator for the Toda system (4.29). We will indicate by the superscript TL the symmetry generators for the Toda Lattice (2.3). With these generators we can associate symmetry generators in the space of the reflection coefficients. These generators are written as

$$\hat{\mathcal{X}}_k^T = \mu\lambda^k R\partial_R. \tag{4.63}$$

In agreement with Lie theory, whenever R is an analytic function of ϵ, the corresponding flows are given by solving the equations

$$\frac{\mathrm{d}\tilde{R}}{\mathrm{d}\epsilon_k} = \mu\lambda^k\tilde{R}, \qquad \frac{\mathrm{d}\lambda}{\mathrm{d}\epsilon_k} = 0, \qquad \tilde{R}(\epsilon_k = 0) = R. \tag{4.64}$$

One can prove that the isospectral symmetry generators (4.62) commute amongst each other

$$\left[\hat{X}_k^T, \hat{X}_m^T\right] = 0, \tag{4.65}$$

by computing the corresponding commutation relation in the space of the reflection coefficients

$$\left[\hat{\mathcal{X}}_k^T, \hat{\mathcal{X}}_m^T\right] = [\mu\lambda^k R\partial_R, \mu\lambda^m R\partial_R] = 0. \tag{4.66}$$

So far, the use of the vector fields in the reflection coefficient space has just reexpressed a known result, namely that the commutation of the reflection coefficients is rewritten as equation (4.66). We now extend the use of vector fields in the space of the spectral data to the case of the nonisospectral symmetries (4.42). Using the definition (4.61) we can introduce the generators of the nonisospectral symmetries for the Toda system. The symmetry vector fields are

$$\begin{aligned}\hat{Y}_k^T = {} & \big\{t\big[\mathcal{L}_{11}^{(k+1)}[a_n(b_n - b_{n+1})] + \mathcal{L}_{12}^{(k+1)}(a_{n-1} - a_n)\big] + \mathcal{L}_{11}^{(k)}[a_n((2n+3)b_{n+1} - (2n-1)b_n)] \\ & + \mathcal{L}_{12}^{(k)}\big[b_n^2 - 4 + 2(n+1)a_n - 2(n-1)a_{n-1}\big]\big\}\partial_{a_n} \\ & + \big\{t\big[\mathcal{L}_{21}^{(k+1)}[a_n(b_n - b_{n+1})] + \mathcal{L}_{22}^{(k+1)}(a_{n-1} - a_n)\big] \\ & + \mathcal{L}_{21}^{(k)}[a_n((2n+3)b_{n+1} - (2n-1)b_n)] \\ & + \mathcal{L}_{22}^{(k)}\big[b_n^2 - 4 + 2(n+1)a_n - 2(n-1)a_{n-1}\big]\big\}\partial_{b_n}. \end{aligned} \tag{4.67}$$

Taking into account equation (4.43), we can define the symmetry generators (4.67) in the space of the spectral data as

$$\hat{\mathcal{Y}}_k^T = \mu\lambda^{k+1} t R \partial_R + \mu^2 \lambda^k \partial_\lambda. \tag{4.68}$$

Commuting $\hat{\mathcal{Y}}_k^T$ with $\hat{\mathcal{Y}}_m^T$ we have

$$\big[\hat{\mathcal{Y}}_k^T, \hat{\mathcal{Y}}_m^T\big] = (m-k)\big[\hat{\mathcal{Y}}_{k+m+1}^T - 4\hat{\mathcal{Y}}_{k+m-1}^T\big]. \tag{4.69}$$

From the relation between the spectral space and the space of the solutions, we conclude that the vector fields representing the symmetries of the studied evolution equations, satisfy the same commutation relations

$$\big[\hat{Y}_k^T, \hat{Y}_m^T\big] = (m-k)\big[\hat{Y}_{k+m+1}^T - 4\hat{Y}_{k+m-1}^T\big]. \tag{4.70}$$

In a similar manner we can work out the commutation relations between the $\hat{Y}_k$ and $\hat{X}_m$ symmetry generators. We get:

$$\big[\hat{\mathcal{X}}_k^T, \hat{\mathcal{Y}}_m^T\big] = -(1+k)\hat{\mathcal{X}}_{k+m+1}^T + 4k\hat{\mathcal{X}}_{k+m-1}^T, \tag{4.71}$$

and consequently

$$\big[\hat{X}_k^T, \hat{Y}_m^T\big] = -(1+k)\hat{X}_{k+m+1}^T + 4k\hat{X}_{k+m-1}^T. \tag{4.72}$$

Relations such as (4.70) and (4.72) can also be checked directly, but the use of the vector field in the reflection coefficient space is much more efficient.

Let us now consider the commutation relations involving the exceptional symmetries (4.44). We write them as

$$\hat{Z}_0^T = \partial_{b_n} \tag{4.73}$$

$$\hat{Z}_1^T = [2a_n + t\dot{a}_n]\partial_{a_n} + [b_n + t\dot{b}_n]\partial_{b_n}. \tag{4.74}$$

As mentioned in section 4.1, these symmetries do not satisfy the asymptotic conditions (4.36). Hence we cannot write down the commutation relations in all generality for all symmetries simultaneously. We calculate explicitly the commutation relations involving just $\hat{Z}_0^T$ and $\hat{Z}_1^T$, $\hat{X}_0^T$, $\hat{X}_1^T$ and $\hat{Y}_0^T$. The nonzero commutation relations are

$$\begin{aligned} &\big[\hat{X}_0^T, \hat{Z}_1^T\big] = -\hat{X}_0^T, && \big[\hat{Z}_0^T, \hat{Z}_1^T\big] = \hat{Z}_0^T && \\ &\big[\hat{Y}_0^T, \hat{Z}_0^T\big] = -2\hat{Z}_1^T, && \big[\hat{Y}_0^T, \hat{Z}_1^T\big] = -\hat{Y}_0^T - 8\hat{Z}_0^T, && \big[\hat{X}_1^T, \hat{Z}_0^T\big] = -2\hat{X}_0^T, \\ &\big[\hat{X}_1^T, \hat{Z}_1^T\big] = -2\hat{X}_1^T, && \big[\hat{X}_0^T, \hat{Y}_0^T\big] = -\hat{X}_1^T, && \big[\hat{X}_1^T, \hat{Y}_0^T\big] = -2\hat{X}_2^T + 4\hat{X}_0^T. \end{aligned} \tag{4.75}$$

In the case of the Toda lattice (2.3) we have (see equations (4.59), (4.58))

$$\hat{Z}_0^{TL} = t\partial_{v_n} \tag{4.76}$$

$$\hat{Z}_1^{TL} = [t\dot{v}_n - 2n]\partial_{v_n} \tag{4.77}$$

and

$$\hat{Z}_{-1}^{TL} = \partial_{v_n} \tag{4.78}$$

in correspondence with equation (4.60). As equations (2.3) and (4.29) are just two different representations of the same system, the symmetry generators in the space of the spectral data are the same. Consequently, the commutation relations between $\hat{X}_n^{TL}$ and $\hat{Y}_m^{TL}$ are given by equations (4.66), (4.70) and (4.72). The symmetries $\hat{X}_0^{TL}$, $\hat{X}_1^{TL}$ and $\hat{Y}_0^{TL}$, according to equations (4.52), (4.53), (4.55) are given by

$$\begin{aligned}
&\hat{X}_0^{TL} = \dot{v}_n \partial_{v_n}, \qquad \hat{X}_1^{TL} = \left[\dot{v}_n^2 + \mathrm{e}^{v_{n-1}-v_n} + \mathrm{e}^{v_n - v_{n+1}} - 2\right]\partial_{v_n} \\
&\hat{Y}_0^{TL} = \left\{t\left[v_{n,t}^2 + \mathrm{e}^{v_{n-1}-v_n} + \mathrm{e}^{v_n-v_{n+1}} - 2\right] - (2n-1)v_{n,t} + w_n(t)\right\}\partial_{v_n} \\
&w_{n+1}(t) - w_n(t) = -2\dot{v}_{n+1}, \qquad \dot{w}_n(t) = 2(\mathrm{e}^{v_n - v_{n+1}} - 1).
\end{aligned} \tag{4.79}$$

The nonzero commutation relations are:

$$\begin{aligned}
&\left[\hat{X}_0^{TL}, \hat{Z}_0^{TL}\right] = -\hat{Z}_{-1}^{TL}, \qquad \left[\hat{X}_0^{TL}, \hat{Z}_1^{TL}\right] = -\hat{X}_0^{TL}, \\
&\left[\hat{X}_0^{TL}, \hat{Y}_0^{TL}\right] = -\hat{X}_1^{TL} + \omega \hat{Z}_{-1}^{TL}, \\
&\left[\hat{X}_1^{TL}, \hat{Z}_0^{TL}\right] = -2\hat{X}_0^{TL}, \qquad \left[\hat{X}_1^{TL}, \hat{Z}_1^{TL}\right] = -2\hat{X}_1^{TL} - 4\hat{Z}_{-1}^{TL}, \\
&\left[\hat{X}_1^{TL}, \hat{Y}_0^{TL}\right] = -2\hat{X}_2^{TL} + 4\hat{X}_0^{TL} + \sigma \hat{Z}_{-1}^{TL}, \\
&\left[\hat{Y}_0^{TL}, \hat{Z}_{-1}^{TL}\right] = \beta \hat{Z}_{-1}^{TL}, \qquad \left[\hat{Y}_0^{TL}, \hat{Z}_0^{TL}\right] = -2\hat{Z}_1^{TL} + \gamma \hat{Z}_{-1}^{TL}, \\
&\left[\hat{Y}_0^{TL}, \hat{Z}_1^{TL}\right] = -\hat{Y}_0^{TL} - 8\hat{Z}_0^{TL} + \delta \hat{Z}_{-1}^{TL}, \qquad \left[\hat{Z}_0^{TL}, \hat{Z}_1^{TL}\right] = \hat{Z}_0^{TL},
\end{aligned} \tag{4.80}$$

where $(\beta, \gamma, \delta, \omega, \sigma)$ are integration constants. The presence of these integration constants indicates that the symmetry algebra of the Toda equation is not completely specified. The constants appear whenever the symmetry $\hat{Y}_0^{TL}$ is involved. The ambiguity is related to the ambiguity in the definition of $\hat{Y}_0^{TL}$ itself, i.e. in the solution of equation (4.79) for $w_n(t)$. We fix these coefficients by requiring that one obtains the correct continuous limit, i.e. in the asymptotic limit, when h goes to zero, a combination of the generators of the Toda Lattice (2.3) and Toda system (4.29) goes over to the symmetry algebra of the potential Korteweg–de Vries equation (see section 4.2.2).

The commutation relations obtained above determine the structure of the infinite-dimensional Lie symmetry algebras. The first symmetry generators are given in equations (4.62), (4.67), (4.73) and (4.74) and the corresponding commutation relations are given by equations (4.70), (4.72) and (4.75). As one can see, the symmetry operators $\hat{Y}_k^T$ and $\hat{Z}_k^T$ are linear in t and the coefficient of t is an isospectral symmetry operator $\hat{X}_k^T$. Consequently, as the operators $\hat{X}_k^T$ commute amongst each other, the commutator of $\hat{X}_m^T$ with any of the $\hat{Y}_k^T$ or $\hat{Z}_k^T$ symmetries will not have any explicit time dependence and thus can be written in terms of $\hat{X}_n^T$ only. Thus the structure of the Lie algebra for the Toda system can be written as:

$$L = L_0 \ni L_1, \qquad L_0 = \left\{\hat{h}, \hat{e}, \hat{f}, \hat{Y}_1^T, \hat{Y}_2^T, \ldots\right\}, \qquad L_1 = \left\{\hat{X}_0^T, \hat{X}_1^T, \ldots\right\} \tag{4.81}$$

where $\left\{\hat{h} = \hat{Z}_1^T, \hat{e} = \hat{Z}_0^T, \hat{f} = \hat{Y}_0^T + 4\hat{Z}_0^T\right\}$ denotes a sl(2, $\mathbf{R}$) subalgebra with $[\hat{h}, \hat{e}] = \hat{e}, [\hat{h}, \hat{f}] = -\hat{f}, [\hat{e}, \hat{f}] = 2\hat{h}$. The algebra L_0 is perfect, i.e. we have $[L_0, L_0] = L_0$. It is worthwhile noting that $\hat{Z}_0^T, \hat{Z}_1^T$ and $\hat{X}_0^T$ are point symmetries while all the others are generalized symmetries. Indeed, all the other vector fields involve other values of the discrete variable than n or time derivatives of the fields.

For the Toda lattice equation the point transformations are $\hat{X}_0^{TL}, \hat{Z}_0^{TL}$ and $\hat{Z}_1^{TL}$, as for the Toda system, plus the additional $\hat{Z}_{-1}^{TL}$. Taking into account equations (4.76)–(4.80), the structure of the Lie algebra is the same as that of the Toda system with $L_0 = \left\{\hat{Z}_{-1}^{TL}, \hat{Z}_0^{TL}, \hat{Z}_1^{TL}, \hat{Y}_0^{TL}, \hat{Y}_1^{TL}, \hat{Y}_2^{TL}, \ldots\right\}$, $L_1 = \left\{\hat{X}_0^{TL}, \hat{X}_1^{TL}, \hat{X}_2^{TL}, \ldots\right\}$.

4.2.2. Contraction of the symmetry algebras in the continuous limit. It is well known [27, 31, 153, 154, 253] that the Toda equation has the potential Korteweg–de Vries equation as one of its possible continuous limits. In fact, by setting

$$v_n(t) = -\tfrac{1}{2}hu(x,\tau) \qquad x = (n-t)h \qquad \tau = -\tfrac{1}{24}h^3 t \tag{4.82}$$

we can write equation (2.3) as

$$\left(u_\tau - u_{xxx} - 3u_x^2\right)_x = \mathcal{O}(h^2) \tag{4.83}$$

i.e. the once differentiated potential Korteweg–de Vries equation. Let us now rewrite the symmetry generators in the new coordinate system defined by (4.82) and develop them for small h in Taylor series. We have

$$\hat{X}_0^{TL} = \left\{-u_x(x,\tau)h - \frac{1}{24}u_\tau(x,\tau)h^3\right\}\partial_u \tag{4.84}$$

$$\hat{X}_1^{TL} = \left\{-2u_x(x,\tau)h - \frac{1}{3}u_\tau(x,\tau)h^3 + \mathcal{O}(h^5)\right\}\partial_u \tag{4.85}$$

$$\hat{X}_2^{TL} = \left\{-4u_x(x,\tau)h - \frac{7}{6}u_\tau(x,\tau)h^3 + \mathcal{O}(h^5)\right\}\partial_u \tag{4.86}$$

$$\hat{Y}_0^{TL} = \{2[u(x,\tau) + xu_x(x,\tau) + 3\tau u_\tau(x,\tau)] + \mathcal{O}(h)\}\partial_u \tag{4.87}$$

$$\hat{Z}_{-1}^{TL} = -\frac{2}{h}\partial_u, \qquad \hat{Z}_0^{TL} = \frac{48}{h^4}\tau\partial_u \tag{4.88}$$

$$\hat{Z}_1^{TL} = \left\{-\frac{96}{h^4}\tau + \frac{4}{h^2}[x + 6\tau u_x(x,\tau)] + \mathcal{O}(1)\right\}\partial_u. \tag{4.89}$$

To obtain equations (4.85)–(4.87) we have used the following evolution for u:

$$u_\tau = u_{xxx} + 3u_x^2. \tag{4.90}$$

The point symmetry generators written in the evolutionary form, for the potential Korteweg–de Vries equation (90) read:

$$\hat{P}_0 = u_\tau\partial_u, \quad \hat{P}_1 = u_x\partial_u, \qquad \hat{B} = [x + 6\tau u_x]\partial_u, \tag{4.91}$$

$$\hat{D} = [u + xu_x + 3\tau u_\tau]\partial_u, \quad \hat{\Gamma} = \partial_u, \tag{4.92}$$

and their commutation table is

	$\hat{P}_0$	$\hat{P}_1$	$\hat{B}$	$\hat{D}$	$\hat{\Gamma}$
$\hat{P}_0$	0	0	$-6\hat{P}_1$	$-3\hat{P}_0$	0
$\hat{P}_1$		0	$-\hat{\Gamma}$	$-\hat{P}_1$	0
$\hat{B}$			0	$2\hat{B}$	0
$\hat{D}$				0	$-\hat{\Gamma}$
$\hat{\Gamma}$					0.

(4.93)

Let us now consider the continuous limit $h \to 0$ of the symmetry algebra of the Toda equation. We can write the simplest symmetry generators of the Toda equation as a linear combination of the generators (4.84)–(4.89), so that in the continuous limit they go over to the generators of the point symmetries of the potential Korteweg–de Vries equations (4.91) and (4.92):

$$\tilde{P}_0 = \frac{4}{h^3}\left(2\hat{X}_0^{TL} - \hat{X}_1^{TL}\right), \qquad \tilde{P}_1 = -\frac{1}{h}\hat{X}_0^{TL}, \qquad \tilde{D} = \frac{1}{2}\hat{Y}_0^{TL}, \tag{4.94}$$

$$\tilde{B} = \frac{h^2}{4}\left(2\hat{Z}_0^{TL} + \hat{Z}_1^{TL}\right), \qquad \tilde{\Gamma} = -\frac{h}{2}\hat{Z}_{-1}^{TL}. \tag{4.95}$$

Taking into account the commutation table between the generators $\hat{X}_0^{TL}$, $\hat{X}_1^{TL}$, $\hat{Z}_{-1}^{TL}$, $\hat{Z}_0^{TL}$, $\hat{Z}_1^{TL}$ and $\hat{Y}_0^{TL}$, given by (4.80) and the continuous limit of $\hat{X}_2^{TL}$ given by equation (4.86), we get:

	$\tilde{P}_0$	$\tilde{P}_1$	$\tilde{B}$	$\tilde{D}$	$\tilde{\Gamma}$
$\tilde{P}_0$	0	0	$-6\tilde{P}_1 + \mathcal{O}(h^2)$	$-3\tilde{P}_0 + \mathcal{O}(h^2)$	0
$\tilde{P}_1$		0	$-\tilde{\Gamma} + \mathcal{O}(h^2)$	$-\tilde{P}_1 + \mathcal{O}(h^2)$	0
$\tilde{B}$			0	$2\tilde{B} + \mathcal{O}(h^2)$	0
$\tilde{D}$				0	$-\tilde{\Gamma}$
$\tilde{\Gamma}$					0.

(4.96)

The results contained in table (4.96) are obtained by setting $\beta = -2, 2\gamma + \delta = 0$ and $\omega = \sigma = 0$. Thus, we have reobtained in the continuous limit, all point symmetries of the potential KdV equation. The limit partially fixes the previously undetermined constants in equation (4.80). To get all point symmetries of the potential KdV equation we needed not only the point symmetries $\hat{X}_0^{TL}$, $\hat{Z}_0^{TL}$, $\hat{Z}_{-1}^{TL}$ and $\hat{Z}_1^{TL}$ of the Toda equation, but also the higher symmetries $\hat{X}_1^{TL}$, $\hat{Y}_0^{TL}$.

This procedure can be viewed as a new application of the concept of Lie algebra contractions. Lie algebra contractions were first introduced by Inönü and Wigner [121] in order to relate the group theoretical foundations of relativistic and nonrelativistic physics. The speed of light c was introduced as a parameter into the commutation relations of the Lorentz group. For $c \to \infty$ the Lorentz group 'contracted' to the Galilei group. Lie algebra contractions thus relate different Lie algebras of the same dimension, but of different isomorphism classes. A systematic study of contractions, relating large families of nonisomorphic Lie algebras of the same dimension, based on Lie algebra grading, was initiated by Moody and Patera [197].

In general Lie algebra and Lie group contractions are extremely useful when describing the mathematical relation between different theories. The contraction parameter can be the Planck constant, when relating quantum systems to classical ones. It can be the curvature k of a space of constant curvature, which for $k \to 0$ goes to a flat space. The contraction will then relate special functions defined e.g. on spheres, to those defined in a Euclidean space [127].

In our case the contraction parameter is the lattice spacing h. Some novel features appear. First of all, we are contracting an infinite-dimensional Lie algebra of generalized symmetries, that of the Toda lattice. The contraction leads to an infinite-dimensional Lie algebra, not isomorphic to the first one. This 'target algebra' is the Lie algebra of point and generalized symmetries of the potential KdV equation. A particularly interesting feature is that the five-dimensional Lie algebra of point symmetries of the potential KdV is obtained from a subset of point and generalized symmetries of the Toda equation. This five-dimensional subset is not an algebra (it is not closed under commutations). It does contract into a Lie algebra in the continuous limit.

4.2.3. Bäcklund transformations for the Toda equation. In addition to symmetry transformations presented in section (4.2.1), the Toda system admits Bäcklund transformations [25, 29, 34, 35, 58, 103, 139, 143, 150]. They are discrete transformations (i.e. mappings) that starting from a solution, produce a new solution. Bäcklund transformations commute amongst each other, allowing the definition of a soliton superposition formula that endows the evolution equation with an integrability feature. Using the spectral transform [35] we can write down families of Bäcklund transformations. They are obtained by requiring the existence of two essentially different solutions to the Lax equations (4.2), (4.3), $\psi(x,t;\lambda)$ and $\tilde{\psi}(x,t;\lambda)$. These two solutions will be associated with two different solutions to the Lax equation (4.6), (4.7), $u(x,t)$ and $\tilde{u}(x,t)$ and consequently two different Lax pairs $(L(u), M(u))$ and $(L(\tilde{u}), M(\tilde{u}))$. If a transformation exists between u and $\tilde{u}$ then there must exist a transformation between $\psi(x,t;\lambda)$ and $\tilde{\psi}(x,t;\lambda)$ and between $(L(u), M(u))$ and $(L(\tilde{u}), M(\tilde{u}))$. This implies that there will exist an operator $D(u,\tilde{u})$, often called the *Darboux operator* which will relate $\tilde{\psi}(x,t;\lambda)$ and $\psi(x,t;\lambda)$, i.e.

$$\tilde{\psi}(x,t;\lambda) = D(u,\tilde{u})\psi(x,t;\lambda). \tag{4.97}$$

Taking into account the Lax equations for $\psi(x,t;\lambda)$ and $\tilde{\psi}(x,t;\lambda)$, we get from (4.97) the following operator equations for D:

$$\tilde{L}(\tilde{u})D(u,\tilde{u}) = D(u,\tilde{u})L(u), \tag{4.98}$$

$$D_t(u,\tilde{u}) = D(u,\tilde{u})M(u) - M(\tilde{u})D(u,\tilde{u}). \tag{4.99}$$

From equations (4.98) and (4.99) we get a class of Bäcklund transformations, which we will symbolically write as $B_j(u(x,t), \tilde{u}(x,t), \ldots)$ characterized by a recursion operator Λ.

In the discrete case, the technique is basically the same, and in the case of the matrix discrete Schrödinger spectral problem, a generalization of the scalar problem (4.31), the appropriate developments are found in [30]. Specializing to the scalar case, the class of Bäcklund transformations associated with the Toda system (4.29) is given by

$$\gamma(\Lambda)\begin{pmatrix}\tilde{a}(n) - a(n)\\ \tilde{b}(n) - b(n)\end{pmatrix} = \delta(\Lambda)\begin{pmatrix}\tilde{\Pi}(n)\Pi^{-1}(n+1)(\tilde{b}(n) - b(n+1))\\ \tilde{\Pi}(n-1)\Pi^{-1}(n) - \tilde{\Pi}(n)\Pi^{-1}(n+1)\end{pmatrix}, \tag{4.100}$$

where $\gamma(z)$ and $\delta(z)$ are entire functions of their argument and we have denoted

$$\Pi(n) = \prod_{j=n}^{\infty} a(j), \qquad \tilde{\Pi}(n) = \prod_{j=n}^{\infty} \tilde{a}(j). \tag{4.101}$$

Above, Λ is the recursion operator

$$\Lambda\begin{bmatrix}p(n)\\ q(n)\end{bmatrix} = \begin{bmatrix} p(n)b(n+1) + \tilde{a}(n)[q(n) + q(n+1)] + \Sigma(n)[\tilde{b}(n) - b(n+1)] \\ \quad +[a(n) - \tilde{a}(n)]\sum_{j=n}^{\infty} p(j) \\ p(n) + \tilde{b}(n)q(n) - \Sigma(n) + \Sigma(n-1) + [b(n) - \tilde{b}(n)]\sum_{j=n}^{\infty} q(j)\end{bmatrix} \tag{4.102}$$

and

$$\Sigma(n) = \tilde{\Pi}(n)\left[\sum_{j=n}^{\infty} \tilde{\Pi}(j)^{-1}p(j)\Pi(j+1)\right]\Pi^{-1}(n+1). \tag{4.103}$$

In [31] it is proven that whenever (a_n, b_n) and $(\tilde{a}_n, \tilde{b}_n)$ satisfy the asymptotic conditions (4.36) and the Bäcklund transformations (4.100), the reflection coefficient satisfies the equation

$$\tilde{R}(\lambda) = \frac{\gamma(\lambda) - \delta(\lambda)z}{\gamma(\lambda) - \delta(\lambda)/z}R(\lambda). \tag{4.104}$$

For the Toda lattice (2.3) the one-soliton Bäcklund transformation, when the functions $\gamma(\lambda)$ and $\delta(\lambda)$ are constant, reads:

$$\dot{\tilde{v}}(n) - \dot{v}(n+1) = \frac{\gamma}{\delta}\{\mathrm{e}^{v(n+1)-\tilde{v}(n+1)} - \mathrm{e}^{v(n)-\tilde{v}(n)}\}. \tag{4.105}$$

Formulae (4.100)–(4.104) also provide much more general transformations, i.e. higher order Bäcklund transformations. If the arbitrary functions $\gamma(\lambda)$ and $\delta(\lambda)$ are polynomials, then we have a finite-order Bäcklund transformation that can be interpreted as a composition of a finite number of one-soliton transformations. In more general cases, when $\gamma(\lambda)$ and $\delta(\lambda)$ are entire functions, we face an infinite-order Bäcklund transformation.

In the following section, we discuss how Bäcklund transformations are related to continuous symmetry transformations, allowing, albeit formally, an integration of the latter.

4.2.4. Relation between Bäcklund transformations and higher symmetries. A general isospectral higher symmetry of the Toda equation is given by

$$\begin{pmatrix} a_{n,\epsilon} \\ b_{n,\epsilon} \end{pmatrix} = \phi(\mathcal{L}) \begin{pmatrix} a_n(b_n - b_{n+1}) \\ a_{n-1} - a_n \end{pmatrix}, \tag{4.106}$$

with the spectrum evolution

$$\frac{\mathrm{d}R(\lambda,\epsilon)}{\mathrm{d}\epsilon} = \mu\phi(\lambda)R(\lambda,\epsilon). \tag{4.107}$$

These equations generalize equation (4.40), used above. In equations (4.106), (4.107) the function ϕ is an entire function of its argument. Equation (4.107) can be formally integrated in the spectral parameter (λ) space, giving

$$R(\lambda,\epsilon) = \mathrm{e}^{\mu\phi(\lambda)}R(\lambda,0). \tag{4.108}$$

Taking into account the following definitions of λ and μ in terms of z,

$$\lambda = \frac{1}{z} + z, \qquad \mu = \frac{1}{z} - z, \qquad \mu^2 = \lambda^2 - 4, \tag{4.109}$$

$$z = \frac{\lambda - \mu}{2}, \qquad \frac{1}{z} = \frac{\lambda + \mu}{2}, \tag{4.110}$$

we can rewrite the general Bäcklund transformation (4.104) as

$$\tilde{R}(\lambda) = \frac{2 - (\lambda - \mu)\beta(\lambda)}{2 - (\lambda + \mu)\beta(\lambda)}R(\lambda), \qquad \beta(\lambda) = \frac{\delta(\lambda)}{\gamma(\lambda)}.$$

In order to identify a general symmetry transformation with a Bäcklund transformation, and vice versa, we equate $R(\lambda,\epsilon) = \tilde{R}(\lambda)$

$$\mathrm{e}^{\mu\phi(\lambda)} = \frac{2 - (\lambda - \mu)\beta(\lambda)}{2 - (\lambda + \mu)\beta(\lambda)} \tag{4.111}$$

and find that $\phi(\lambda)$ in equation (4.108) is given by

$$\phi(\lambda) = \frac{1}{\mu}\ln\left[\frac{2 - (\lambda - \mu)\beta(\lambda)}{2 - (\lambda + \mu)\beta(\lambda)}\right]. \tag{4.112}$$

The right-hand side of equation (4.112) must not depend on μ. Relations (4.109) allow us to separate the exponential in (4.111) into two entire components $E_0(\lambda)$ and $E_1(\lambda)$ satisfying

$$\mathrm{e}^{\mu\phi(\lambda)} = \cosh[\mu\phi(\lambda)] + \mu\frac{\sinh[\mu\phi(\lambda)]}{\mu} = E_0(\lambda) + \mu E_1(\lambda). \tag{4.113}$$

Noting that the rhs of equation (4.112) is rational in μ, developing μ^2 and identifying powers (0th and 1st) of μ, we get a system of two compatible equations

$$-(2 - \lambda\beta)E_0 + (\lambda^2 - 4)\beta E_1 = -(2 - \lambda\beta), \tag{4.114}$$

$$-\beta E_0 + (2 - \lambda\beta)E_1 = \beta. \tag{4.115}$$

Equations (4.114), (4.115) provide us with explicit formulae relating a given general higher symmetry (characterized by ϕ, and thus E_0, E_1) with a general Bäcklund transformation (characterized by γ and δ, and thus by β):

$$\beta(\lambda) = \frac{\delta(\lambda)}{\gamma(\lambda)} = \frac{2E_1}{E_0 + \lambda E_1 + 1} = \frac{2\sinh[\mu\phi(\lambda)]/\mu}{\cosh[\mu\phi(\lambda)] + \lambda\sinh[\mu\phi(\lambda)]/\mu + 1}. \tag{4.116}$$

From this equation we see that whatsoever be the symmetry, we find a Bäcklund transformation, i.e. for an arbitrary function ϕ we obtain the two entire functions γ and δ. Vice versa, given a general Bäcklund transformation, we can find a corresponding generalized symmetry

$$E_0 = -\frac{2(\beta^2 - 1) + \lambda\beta(2 - \lambda\beta)}{2(\beta^2 - \lambda\beta + 1)}, \qquad E_1 = -\frac{(\lambda\beta - 2)\beta}{2(\beta^2 - \lambda\beta + 1)}, \tag{4.117}$$

or more explicitly,

$$\phi(\lambda) = \frac{1}{\mu}\sinh^{-1}\left[-\mu\frac{(\lambda\beta - 2)\beta}{2(\beta^2 - \lambda\beta + 1)}\right]. \tag{4.118}$$

In the case of a one-soliton Bäcklund transformation with $\beta = 1$, we have:

$$E_0 = -\frac{\lambda}{2}, \qquad E_1 = \frac{1}{2}$$

and we can write $\phi(\lambda)$ as

$$\phi(\lambda) = \frac{\sinh^{-1}[\sqrt{\lambda^2 - 4}/2]}{\sqrt{\lambda^2 - 4}}. \tag{4.119}$$

In this simple case we can write the symmetry in a closed form as an infinite sequence of elementary symmetry transformations:

$$\phi(\lambda) = \sum_{k=0}^{\infty}\left[\frac{(2k)!\pi}{k!(k-1)!2^{4k+2}}\lambda^{2k} + \frac{1}{2}\frac{k!(k+1)!}{(2k+2)!}\lambda^{2k+1}\right]. \tag{4.120}$$

In this way, the existence of a one-soliton transformation implies the existence of an infinite-order generalized symmetry.

Let us consider the time shift symmetry given by $\phi(\lambda) = 1$. Then equation (4.113) implies that $E_0 = \cosh\mu$ and $E_1 = \sinh\mu/\mu$. According to (4.116) the corresponding higher Bäcklund transformation is

$$\delta(\lambda) = 2\sinh\mu/\mu \tag{4.121}$$

$$\gamma(\lambda) = \cosh\mu + \lambda\sinh\mu/\mu + 1. \tag{4.122}$$

This Bäcklund transformation, corresponding to the point symmetry studied, is of infinite order.

5. Lie point symmetries of difference schemes

In this section we take a point of view complementary to that of sections 2–4. We restrict ourselves to continuous point symmetries only. On the other hand, we shall consider flexible difference schemes, with group transformations acting on the entire scheme. We make full use of the formalism presented in section 1.2 of the introduction.

The difference scheme, i.e. the difference equation and the lattice are described by the n_E equations (1.18). The group transformations transform solutions of this scheme amongst each other.

We shall consider ordinary and partial difference schemes separately.

5.1. Lie point symmetries of ordinary difference schemes

As stated in the introduction, an OΔS will consist of two equations of the form (1.18). For instance, a 3-point scheme can be written as

$$E_a(x_{n-1}, x_n, x_{n+1}, u_{n-1}, u_n, u_{n+1}) = 0, \qquad a = 1, 2 \tag{5.1}$$

satisfying equation (1.19) with $M = 1$, $N = -1$ (possibly after an upshift or downshift of n).

When taking the continuous limit it is convenient to introduce different quantities, namely differences between neighbouring points and discrete derivatives like

$$\begin{aligned} h_+ &= x_{n+1} - x_n, & h_- &= x_n - x_{n-1}, \\ u_x &= \frac{u_{n+1} - u_n}{x_{n+1} - x_n}, & u_{\underline{x}} &= \frac{u_n - u_{n-1}}{x_n - x_{n-1}}, \\ u_{x\underline{x}} &= 2\frac{u_{,x} - u_{,\underline{x}}}{x_{n+1} - x_{n-1}}, \ldots . \end{aligned} \tag{5.2}$$

In the continuous limit, we have

$$h_+ \to 0, \qquad h_- \to 0, \qquad u_x \to u', \quad u_{\underline{x}} \to u', \qquad u_{x\underline{x}} \to u''.$$

As a clarifying example of the meaning of the difference scheme (1.18), let us consider a three-point scheme that will approximate a second-order linear difference equation:

$$E_1 = \frac{u_{n+1} - 2u_n + u_{n-1}}{(x_{n+1} - x_n)^2} - u_n = 0, \tag{5.3}$$

$$E_2 = x_{n+1} - 2x_n + x_{n-1} = 0. \tag{5.4}$$

The solution of equation $E_2 = 0$ determines a uniform lattice

$$x_n = hn + x_0. \tag{5.5}$$

The scale h and the origin x_0 in equation (5.5) are not fixed by equation (5.4), instead they appear as integration constants, i.e., they are *a priori* arbitrary. Once they are chosen, equation (5.3) reduces to a linear difference equation with constant coefficients, since we have $x_{n+1} - x_n = h$. Thus, a solution of equation (5.3) will have the form

$$u_n = \lambda^{x_n}. \tag{5.6}$$

Substituting (5.6) into (5.3) we obtain the general solution of the difference scheme (5.3), (5.4),

$$\begin{aligned} u(x_n) &= c_1\lambda_1^{x_n} + c_2\lambda_2^{x_n}, \qquad x_n = hn + x_0, \\ \lambda_{1,2} &= \left(\frac{2 + h^2 \pm h\sqrt{4 + h^2}}{2}\right)^{1/2}. \end{aligned} \tag{5.7}$$

The solution (5.7) of system (5.3)–(5.4) depends on four arbitrary constants c_1, c_2, h and x_0.

Now let us consider a general three-point scheme of the form (5.1) The two conditions on the Jacobians (1.19) are sufficient to allow us to calculate (x_{n+1}, u_{n+1}) if $(x_{n-1}, u_{n-1}, x_n, u_n)$ are known. Similarly, (x_{n-1}, u_{n-1}) can be calculated if $(x_n, u_n, x_{n+1}, u_{n+1})$ are known. The general solution of the scheme will hence depend on four arbitrary constants and will have the form

$$u_n = f(x_n, c_1, c_2, c_3, c_4) \tag{5.8}$$

$$x_n = \phi(n, c_1, c_2, c_3, c_4). \tag{5.9}$$

A treasury of information on difference equations and their solutions can be found in many classical books [70, 130, 196, 255].

Here we shall follow rather closely the article [158] and use the formalism outlined in section 1.2.

As in the case of differential equations, our basic tool will be vector fields of the form (1.4). In the case of OΔS they will have the form

$$\hat{X} = \xi(x,u)\partial_x + \phi(x,u)\partial_u, \tag{5.10}$$

with

$$x \equiv x_n, \qquad u \equiv u_n = u(x_n).$$

Because we are considering point transformations, ξ and ϕ in (5.10) depend on x and u at one point only.

The prolongation of the vector field $\hat{X}$ is as in equation (1.20), i.e. we prolong to all points of the lattice figuring in scheme (1.18). In these terms the requirement that the transformed function $\tilde{u}(\tilde{x})$ and the variable $\tilde{x}$ should satisfy the same OΔS as the original $u(x)$ is expressed by the requirement

$$\mathrm{pr}\,\hat{X} E_a|_{E_1=E_2=0} = 0, \qquad a = 1, 2. \tag{5.11}$$

Since we must respect both the difference equation and the lattice, we have two conditions (5.11) from which to determine $\xi(x,u)$ and $\phi(x,u)$. Since each of these functions depends on a single point (x,u) and the prolongation (1.20) introduces $N - M + 1$ points in space $X \times U$, equation (5.11) will imply a system of determining equations for ξ and ϕ. Moreover, in general this will be an overdetermined system of linear functional equations that we transform into an overdetermined system of linear partial differential equations [7, 8].

Let us first of all check that the prolongations (1.20) have the correct continuous limit. We consider the first prolongation in the discrete case

$$\begin{aligned} &\mathrm{pr}^1\hat{X} = \xi(x,u)\partial_x + \phi(x,u)\partial_u + \xi(x_+,u_+)\partial_{x_+} + \phi(x_+,u_+)\partial_{u_+}, \\ &x_+ \equiv x_{n+1}, \qquad u_+ \equiv u_{n+1}, \end{aligned} \tag{5.12}$$

and apply it to a function of the variables x, u, h_+, u_x (5.2). The continuous limit is recovered by taking $h \to 0$ and putting

$$\begin{aligned} &u_+ = u(x_+) = u(x) + hu'(x) + \cdots, \\ &\xi(x_+,u_+) = \xi(x,u) + h(\xi_{,x} + \xi_{,u}u') + \cdots, \\ &\phi(x_+,u_+) = \phi(x,u) + h(\phi_{,x} + \phi_{,u}u') + \cdots, \end{aligned} \tag{5.13}$$

where u' is the (continuous) derivative of $u(x)$. We have

$$\begin{aligned} \mathrm{pr}^1_D\hat{X}F(x,u,h,u_x) = \Big\{&\xi\partial_x + \phi\partial_u + [\xi_{,x} + \xi_{,u}u']h\partial_h \\ &+ [-u'(\xi_{,x} + \xi_{,u}u') + \phi_{,x} + \phi_{,u}u']\partial_{u_x} + \cdots\Big\}F. \end{aligned}$$

In the limit when $h \to 0$ we obtain

$$\lim_{h\to 0}\mathrm{pr}^1_D\hat{X} = \xi(x,u)\partial_x + \phi(x,u)\partial_u + \phi^x(x,u,u')\partial_{u'}, \tag{5.14}$$

$$\phi^x = \phi_{,x} + (\phi_{,u} - \xi_{,x})u' - \xi_{,u}u'^2, \tag{5.15}$$

which is the correct expression for the first prolongation in the continuous case [211]. The proof that the nth prolongation (1.20) has the correct continuous limit can be performed by induction.

5.2. Examples

5.2.1. Power nonlinearity on a uniform lattice. Let us consider a difference scheme that is a discretization of the ODE

$$u'' - u^N = 0, \qquad N \neq 0, 1. \tag{5.16}$$

For $N \neq -3$, equation (5.16) is invariant under a two-dimensional Lie group whose Lie algebra is given by

$$\hat{X}_1 = \partial_x, \qquad \hat{X}_2 = (N-1)x\partial_x - 2u\partial_u \tag{5.17}$$

(translations and dilations). For $N = -3$ the symmetry algebra is three-dimensional, isomorphic to $\mathrm{sl}(2, \mathbb{R})$, i.e., it contains a third element in addition to (5.17). A convenient basis for the symmetry algebra of the equation

$$u'' - u^{-3} = 0 \tag{5.18}$$

is

$$\hat{X}_1 = \partial_x, \qquad \hat{X}_2 = 2x\partial_x + u\partial_u, \qquad \hat{X}_3 = x(x\partial_x + u\partial_u). \tag{5.19}$$

A very natural OΔS that has (5.16) as its continuous limit is

$$E_1 = \frac{u_{n+1} - 2u_n + u_{n-1}}{(x_{n+1} - x_n)^2} - u_n^N = 0 \qquad N \neq 0, 1 \tag{5.20}$$

$$E_2 = x_{n+1} - 2x_n + x_{n-1} = 0. \tag{5.21}$$

Let us now apply the symmetry algorithm described in section 5.1 to system (5.20)–(5.21). To illustrate the method, we shall present (just once) all calculations in detail.

First, we choose two variables that will be substituted into equation (5.11), once the prolonged vector field (1.20) is applied to system (5.20)–(5.21), namely

$$\begin{aligned} x_{n+1} &= 2x_n - x_{n-1} \\ u_{n+1} &= (x_n - x_{n-1})^2 u_n^N + 2u_n - u_{n-1}, \end{aligned} \tag{5.22}$$

where, $x_n, u_n x_{n-1}, u_{n-1}$ are our independent variables. By applying $\mathrm{pr}\,\hat{X}$ of (5.10) to equation (5.22) we get:

$$\xi = b = b_1 x + b_0, \tag{5.23}$$

where b_0 and b_1 are constants. To obtain the function $\phi(x_n, u_n)$, we apply $\mathrm{pr}X$ to equation (5.20) and obtain

$$\phi = \phi_1 u + \phi_0(x), \qquad \phi_1 = \mathrm{const} \tag{5.24}$$

where $\phi_0(x)$ satisfies the equation

$$\begin{aligned} &\phi_0(2x_n - x_{n-1}) - 2\phi_0(x_n) + \phi_0(x_{n-1}) - (x_n - x_{n-1})^2((N-1)\phi_1 + 2b_1)u_n^N \\ &\quad - N(x_n - x_{n-1})^2 \phi_0 u_n^{N-1} = 0. \end{aligned} \tag{5.25}$$

We have $N \neq 0, 1$ and hence (5.25) implies that

$$\phi_0 = 0, \qquad (N-1)\phi_1 + 2b_1 = 0. \tag{5.26}$$

We have thus proven that the symmetry algebra of the OΔS (5.20)–(5.21) is the same as that of the ODE (5.16), namely the algebra (5.17).

We observe that the value $N = -3$ is not distinguished here and that system (5.20)–(5.21) is not invariant under $SL(2, \mathbb{R})$ for $N = -3$. Actually, a difference scheme invariant under $SL(2, \mathbb{R})$ does exist and it will have equation (5.18) as its continuous limit. It will not however have the form (5.20)–(5.21), and the lattice will not be uniform [66, 67]. The corresponding $SL(2, R)$ invariant scheme is presented below in section 5.3.3.

Had we taken a two-point lattice, $x_{n+1} - x_n = h$ with h fixed, instead of $E_2 = 0$ as in (5.21), we would only have obtained translational invariance for the equation (5.20) and lost the dilational invariance represented by X_2 of equation (5.17).

5.2.2. An $O\Delta S$ involving an arbitrary function on a uniform lattice. We consider

$$E_1 = \frac{u_{n+1} - 2u_n + u_{n-1}}{(x_{n+1} - x_n)^2} - f\left(\frac{u_n - u_{n-1}}{x_n - x_{n-1}}\right) = 0, \tag{5.27}$$

$$E_2 = x_{n+1} - 2x_n + x_{n-1} = 0, \tag{5.28}$$

where $f(z)$ is some sufficiently smooth function satisfying

$$f''(z) \neq 0. \tag{5.29}$$

The continuous limit of equations (5.27) and (5.28) is

$$u'' - f(u') = 0, \tag{5.30}$$

and it is invariant under a two-dimensional group with Lie algebra,

$$\hat{X}_1 = \partial_x, \qquad \hat{X}_2 = \partial_u, \tag{5.31}$$

for any function $f(u')$. For certain functions f the symmetry group is three-dimensional, where the additional basis element of the Lie algebra is

$$\hat{X}_3 = (ax + bu)\partial_x + (cx + \mathrm{d}u)\partial_u. \tag{5.32}$$

Now let us consider the discrete system (5.27)–(5.28). Before applying pr $\hat{X}$ to this system we choose two variables to substitute into equation (5.11), namely x_{n+1} and u_{n+1}. Applying pr $\hat{X}$ to equation (5.28) and then to (5.27) and performing the same kind of passages necessary to solve the determining equation as we did for equation (5.22) we get

$$\xi = \alpha x + \beta, \tag{5.33}$$

with $\alpha = \text{const}$, $\beta = \text{const}$ and

$$\begin{aligned}\phi(x + h, u + hz + h^2 f(z)) - 2\phi(x, u) + \phi(x - h, u - hz)\\ = 2\alpha h^2 f(z) + h^2 f'(z)\left(\frac{\phi(x, u) - \phi(x - h, u - hz)}{h} - \alpha z\right)\end{aligned} \tag{5.34}$$

where we have defined

$$z = \left[\frac{u_n - u_{n-1}}{x_n - x_{n-1}}\right], \qquad h = x_{n+1} - x_n. \tag{5.35}$$

In general, equation (5.34) is quite difficult to solve and for most functions $f(z)$ it has no other solutions than those corresponding to (5.31).

However, if we make the choice

$$f(z) = \mathrm{e}^{-z} \tag{5.36}$$

we find that equation (5.34) is solved by putting

$$\alpha = 1, \qquad \phi = x + u. \tag{5.37}$$

Finally, we find that the system

$$\frac{u_{n+1} - 2u_n + u_{n-1}}{(x_{n+1} - x_n)^2} = \exp\left(\frac{u_n - u_{n-1}}{x_n - x_{n-1}}\right), \qquad x_{n+1} - 2x_n + x_{n-1} = 0 \tag{5.38}$$

is left invariant by a three-dimensional transformation group, generated by the solvable Lie algebra with basis

$$\hat{X}_1 = \partial_x, \qquad \hat{X}_2 = \partial_u, \qquad \hat{X}_3 = x\partial_x + (x + u)\partial_u. \tag{5.39}$$

For further examples see the original article [158] and the lectures [264].

5.3. Symmetry preserving discretization of ordinary differential equations

5.3.1. General comments. In sections 5.1 and 5.2 we assumed that an OΔS (1.18) is given and we showed how to determine its symmetries.

Here we will discuss a different problem, namely the construction of OΔS with *a priori* given symmetry groups. More specifically, we start from a given ODE

$$E(x, y, \dot{y}, \ddot{y}, \ldots) = 0 \tag{5.40}$$

and its symmetry algebra, realized by vector fields of the form (5.10). We now wish to construct an OΔS (1.18), approximating the ODE (5.40) and having the same symmetry algebra (and the same symmetry group).

This can be done systematically, once the order of the ODE (5.40) is fixed. In general, the motivation for such a study is multifold. In physical applications the symmetry may actually be more important than the equation itself. A discrete scheme with the correct symmetries has a good chance of describing the physics correctly. This is specially true if the underlying phenomena really are discrete and the differential equations come from a continuum approximation. Furthermore, the existence of point symmetries for differential and difference equations makes it possible to obtain explicit analytical solutions. Finally, a discretization respecting point symmetries should provide improved numerical methods.

Let us first outline the general method of discretization. If the ODE (5.40) is of order N we need an OΔS involving at least $N + 1$ points, i.e. $N + 1$ pairs

$$\{x_i, u_i; i = 1, \ldots, N + 1\}. \tag{5.41}$$

The procedure is as follows

(1) Take the Lie algebra $\mathfrak{g}$ of the symmetry group $\mathfrak{G}$ of the ODE (5.40) and prolong the (known) vector fields $\{\hat{X}_1, \ldots, \hat{X}_n\}$ to all $N + 1$ points (5.41), as in equation (1.20).

(2) Find a basis for all invariants of the (prolonged) Lie algebra $\mathfrak{g}$ in the space (5.41) of independent and dependent variables. Such a basis will consist of K functionally independent invariants

$$I_a = I_a(x_1, \ldots, x_{N+1}, u_1, \ldots, u_{N+1}), \qquad 1 \leqslant a \leqslant K. \tag{5.42}$$

They are determined by solving the differential equations

$$\mathrm{pr}\, \hat{X}_i I_a(x_1, \ldots, x_{N+1}, u_1, \ldots, u_{N+1}) = 0, \qquad i = 1, \ldots, n. \tag{5.43}$$

Alternatively, the invariants can be found using the moving frame method developed by Olver and collaborators [24, 212–214]. The actual number K satisfies

$$K = 2N + 2 - (\dim \mathfrak{g} - \dim \mathfrak{g}_0) \tag{5.44}$$

where $\mathfrak{g}_0$ is the Lie algebra of the subgroup $\mathfrak{G}_0 \subset \mathfrak{G}$, stabilizing the $N + 1$ points (5.41). We need at least two independent invariants of the form (5.42) to write an invariant OΔS.

(3) If the number of invariants is not sufficient, we can make use of invariant manifolds. To find them, we first write out the matrix of coefficients of the prolonged vector fields $\{\hat{X}_1, \ldots, \hat{X}_n\}$:

$$M = \begin{pmatrix} \xi_{11} & \xi_{12} & \ldots & \xi_{1N+1} & \phi_{11} & \phi_{12} & \ldots & \phi_{1N+1} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \xi_{n1} & \xi_{n2} & \ldots & \xi_{nN+1} & \phi_{n1} & \phi_{n2} & \ldots & \phi_{nN+1} \end{pmatrix} \tag{5.45}$$

and determine the manifolds on which the rank of M, $\mathrm{Rank}(M)$, satisfies

$$\mathrm{Rank}(M) < \min(n, 2N + 2), \tag{5.46}$$

i.e. is less than maximal. The invariant manifolds are then obtained by requiring that equation (5.43) be satisfied on the manifold satisfying equation (5.46).

5.3.2. Symmetries of second-order ODEs. Let us now restrict to the case of a second-order ODE

$$\ddot{u} = F(x, u, \dot{u}). \tag{5.47}$$

Sophus Lie gave a symmetry classification of second-order ODE's (over the field of complex numbers $\mathcal{C}$) [180, 181]. A similar classification over $\mathcal{R}$ is much more recent [190, 191].

The main classification results can be summed up as follows.

(1) The dimension $n = \dim \mathfrak{g}$ of the symmetry algebra of equation (5.47) can be $\dim \mathfrak{g} = 0, 1, 2, 3$ or 8.

(2) If we have $\dim \mathfrak{g} = 1$ we can decrease the order of equation (5.47) by 1. If the dimension is $\dim \mathfrak{g} \geqslant 2$ we can integrate by quadratures.

(3) If we have $\dim \mathfrak{g} = 8$, then the symmetry algebra is $sl(3, \mathcal{C})$, or $sl(3, \mathcal{R})$, respectively. The equation can be transformed into $\ddot{y} = 0$ by a point transformation.

Further symmetry results are due to Noether [208] and Bessel–Hagen [18]. Every ODE (5.47) can be interpreted as an Euler–Lagrange equation for some Lagrangian density

$$\mathcal{L} = \mathcal{L}(x, u, \dot{u}). \tag{5.48}$$

The equation is

$$\frac{\partial \mathcal{L}}{\partial u} - D\left(\frac{\partial \mathcal{L}}{\partial \dot{u}}\right) = 0, \qquad D = \frac{\partial}{\partial x} + \dot{u}\frac{\partial}{\partial u} + \ddot{u}\frac{\partial}{\partial \dot{u}} + \cdots. \tag{5.49}$$

An *infinitesimal divergence symmetry*, or a Lagrangian symmetry is a vector field $\hat{X}$ (5.10) satisfying

$$\mathrm{pr}\, \hat{X}(\mathcal{L}) + \mathcal{L} D(\xi) = D(V), \qquad V = V(x, u), \tag{5.50}$$

where V is some local function of x and u. A symmetry of the Lagrangian $\mathcal{L}$ is always a symmetry of the Euler–Lagrange equation (5.49); however, equation (5.49) may have additional, non-Lagrangian symmetries.

A relevant symmetry result is that if we have $\dim \mathfrak{g} = 1$, or $\dim \mathfrak{g} = 2$ for equation (5.47), then there always exists a Lagrangian having the same symmetry. For $\dim \mathfrak{g} = 3$, at least a two-dimensional subalgebra of the Lagrangian symmetries exists. For $\dim \mathfrak{g} = 8$ a four-dimensional solvable subalgebra of Lagrangian symmetries exists.

Once a Lagrangian (5.48) is found for which equation (5.47) is the Euler–Lagrange equation, every Lagrangian symmetry can be used to find a first integral. Indeed, Noether's theorem tells us that if $\hat{X}$ of equation (5.10) is a Lagrangian symmetry of equation (5.47), then K defined by

$$\xi \mathcal{L} + (\eta - \xi u')\frac{\mathrm{d}\mathcal{L}}{\mathrm{d}u'} - V = K \tag{5.51}$$

is a first integral of this equation.

If we have two such first integrals K_1 and K_2, we can eliminate the derivative u' from them and thus obtain the general solution of the ODE. For differential equations this is not particularly useful, since it is easier to use the symmetries to integrate directly. However, this Lagrangian integration procedure is the one that generalizes to difference systems (see section 5.3.4).

5.3.3. Symmetries of the three-point difference schemes. A symmetry classification of three-point difference schemes was performed quite recently [66, 67]. It is similar to Lie's classification of second-order ODE's and goes over into this classification in the continuous limit. We shall now review the main results of the classification following the method outlined in section 5.3.1.

Sophus Lie [182] gave a classification of all finite-dimensional Lie algebras that can be realized by vector fields of the form (5.10). This was done over the field $\mathbb{C}$ and thus amounts to a classification of finite-dimensional subalgebras of diff$(2, \mathbb{C})$, the Lie algebra of the group of diffeomorphisms of the complex plane $\mathbb{C}^2$. A similar classification of finite-dimensional subalgebras of diff$(2, \mathbb{R})$ exists [93], but we restrict ourselves to the simpler complex case.

For the sake of brevity, we introduce the following notation for three neighbouring points on the lattice:

$$\begin{aligned} x_- &= x_{n-1}, & x &= x_n, & x_+ &= x_{n+1}, \\ u_- &= u_{n-1}, & u &= u_n, & u_+ &= u_{n+1}. \end{aligned} \tag{5.52}$$

Let us now proceed by dimension of the symmetry algebras.

dim $\mathfrak{g}$ = 1. A single vector field can always be rectified into the form

$$A_{1,1}: \quad \hat{X}_1 = \frac{\partial}{\partial u}. \tag{5.53}$$

The invariant ODE is

$$\ddot{u} = F(x, \dot{u}). \tag{5.54}$$

Putting $\dot{u} = y$ we obtain a first-order ODE.

The difference invariants of X_1 are

$$x, h_+ = x_+ - x, \qquad h_- = x - x_-, \qquad \eta_+ = u_+ - u, \qquad \eta_- = u - u_-. \tag{5.55}$$

Using these invariants and the notation (5.2), we can write a difference scheme

$$u_{x\underline{x}} = F\left(x, \frac{u_x + u_{\underline{x}}}{2}, h_-\right), \qquad h_+ = h_- G\left(x, \frac{u_x + u_{\underline{x}}}{2}, h_-\right). \tag{5.56}$$

This scheme goes into equation (5.54) if we require that the otherwise arbitrary functions F and G satisfy

$$\lim_{h_- \to 0} F\left(x, \frac{u_x + u_{\underline{x}}}{2}, h_-\right) = F(x, \dot{u}), \qquad \lim_{h_- \to 0} G\left(x, \frac{u_x + u_{\underline{x}}}{2}, h_-\right) < \infty. \tag{5.57}$$

dim $\mathfrak{g}$ = 2. Precisely four equivalence classes of two-dimensional subalgebras of diff$(2, \mathbb{C})$ exist. Let us consider them separately.

$A_{2,1}$

$$\hat{X}_1 = \partial_x, \qquad \hat{X}_2 = \partial_u. \tag{5.58}$$

The algebra $A_{2,1}$ is Abelian, the elements X_1 and X_2 are linearly nonconnected (linearly independent at any point (x, y)). The invariant ODE is

$$\ddot{u} = F(\dot{u}), \tag{5.59}$$

and can be immediately integrated.

An invariant difference scheme is given by any two relations between the invariants h_+, h_-, η_+, η_- of equation (5.55), for instance

$$u_{x\underline{x}} = F\Big(\frac{u_x + u_{\underline{x}}}{2}, h_-\Big), \qquad h_+ = h_- G\Big(\frac{u_x + u_{\underline{x}}}{2}, h_-\Big), \tag{5.60}$$

with conditions (5.57) imposed on the functions F and G.

$A_{2,2}$

$$\hat{X}_1 = \partial_u, \qquad \hat{X}_2 = x\partial_x + u\partial_u \tag{5.61}$$

This Lie algebra in non-Abelian, the two elements are linearly nonconnected. The invariant ODE is

$$\ddot{u} = \frac{1}{x}F(\dot{u}). \tag{5.62}$$

A basis for the difference invariants is

$$\left\{xu_{x\underline{x}}, u_x + u_{\underline{x}}, \frac{h_+}{h_-}, \frac{h_-}{x}\right\} \tag{5.63}$$

so a possible invariant difference scheme is

$$u_{x\underline{x}} = \frac{1}{x}F\left(\frac{u_x + u_{\underline{x}}}{2}, \frac{h_-}{x}\right), \qquad h_+ = h_- G\left(\frac{u_x + u_{\underline{x}}}{2}, \frac{h_-}{x}\right). \tag{5.64}$$

$A_{2,3}$

$$\hat{X}_1 = \partial_u, \qquad \hat{X}_2 = x\partial_u. \tag{5.65}$$

The algebra is Abelian, the elements $\hat{X}_1$ and $\hat{X}_2$ are linearly connected. The invariant ODE is

$$\ddot{u} = F(x). \tag{5.66}$$

This equation is linear and hence has an eight-dimensional symmetry algebra isomorphic to $SL(2, \mathbb{C})$ (of which $A_{2,3}$ is just a subalgebra).

The difference invariants are

$$\{u_{x\bar{x}}, x, h_+, h_-\} \tag{5.67}$$

so the invariant difference scheme will also be linear (at least in the dependent variable u).

$A_{2,4}$

$$\hat{X}_1 = \partial_u, \qquad \hat{X}_2 = u\partial_u \tag{5.68}$$

The algebra is non-Abelian and isomorphic to $A_{2,2}$, but with linearly connected elements. The invariant ODE is again linear,

$$\ddot{u} = F(x)\dot{u}, \tag{5.69}$$

as is the invariant difference scheme. Equation (5.69) is invariant under the group $SL(3, \mathbb{C})$. Difference invariants are

$$\left\{\xi = 2\frac{u_{x\underline{x}}}{u_x + u_{\underline{x}}}, x, h_+, h_-\right\} \tag{5.70}$$

and a possible invariant OΔS is

$$2\frac{u_{x\underline{x}}}{u_x + u_{\underline{x}}} = F(x, h_-), \qquad G(x, h_+, h_-) = 0. \tag{5.71}$$

dim $\mathfrak{g}$ = 3. We now turn to difference schemes invariant under three-dimensional symmetry groups. We will restrict ourselves to the case when the corresponding ODE is nonlinear. Hence we will omit all algebras that contain $A_{2,3}$ or $A_{2,4}$ subalgebras (they were considered in [66]).

$A_{3,1}$

$$\hat{X}_1 = \partial_x, \qquad \hat{X}_2 = \partial_u, \qquad \hat{X}_3 = x\partial_x + ku\partial_u, \qquad k \neq 0, \tfrac{1}{2}, 1, 2. \tag{5.72}$$

The invariant ODE is

$$\ddot{u} = \dot{u}^{\frac{k-2}{k-1}}. \tag{5.73}$$

For $k = 1$ there is no invariant second-order equation; for $k = 2$ the equation is linear, for $k = \frac{1}{2}$ it is trasformable into a linear equation and has a larger symmetry group (namely $SL(3, \mathbb{C})$.

Difference invariants are

$$I_1 = \frac{h_+}{h_-}, \qquad I_2 = u_x h_+^{1-k}, \qquad I_3 = u_{\underline{x}} h_-^{1-k}. \tag{5.74}$$

A simple invariant difference scheme is

$$u_{x\underline{x}} = \left(\frac{u_x + u_{\underline{x}}}{2}\right)^{[\frac{k-2}{k-1}]} f\left(\frac{u_x + u_{\underline{x}}}{2} h_-^{1-k}\right), \qquad h_+ = h_- g\left(\frac{u_x + u_{\underline{x}}}{2} h_-^{1-k}\right). \tag{5.75}$$

We shall see below that other invariant schemes may be more convenient.

$A_{3,2}$

$$\hat{X}_1 = \partial_x, \qquad \hat{X}_2 = \partial_u, \qquad \hat{X}_3 = x\partial_x + (x + u)\partial_u. \tag{5.76}$$

The invariant ODE is

$$\ddot{u} = \mathrm{e}^{-\dot{u}}. \tag{5.77}$$

Difference invariants in this case are

$$I_1 = \frac{h_+}{h_-}, \qquad I_2 = h_+ \mathrm{e}^{-u_x}, \qquad I_3 = h_- \mathrm{e}^{-u_{\underline{x}}}. \tag{5.78}$$

A possible invariant scheme is

$$u_{x\underline{x}} = \mathrm{e}^{-\frac{u_x + u_{\underline{x}}}{2}} f\left(\sqrt{h_- h_+} \mathrm{e}^{-\frac{u_x + u_{\underline{x}}}{2}}\right), \qquad h_+ = h_- g\big(\sqrt{h_- h_+} \mathrm{e}^{-\frac{u_x + u_{\underline{x}}}{2}}\big). \tag{5.79}$$

No further solvable three-dimensional subalgebras of diff$(2, \mathbb{C})$ exist (though there is another family for diff$(2, \mathbb{R})$ [66]).

Two inequivalent realizations of $sl(2, \mathbb{C})$ leading to second order invariant ODES exist. Let us consider them separately.

$A_{3,3}$

$$\hat{X}_1 = \partial_x, \qquad \hat{X}_2 = 2x\partial_x + u\partial_u, \qquad \hat{X}_3 = x^2\partial_x + xu\partial_u. \tag{5.80}$$

The corresponding invariant ODE is

$$\ddot{u} = u^{-3}, \tag{5.81}$$

and its general solution is

$$u^2 = A(x - x_0)^2 + \frac{1}{A}, \qquad A \neq 0. \tag{5.82}$$

A convenient set of difference invariants is

$$\begin{aligned} I_1 &= \frac{h_+}{uu_+}, & I_2 &= \frac{h_-}{h_+ + h_-} \frac{u_+}{u}, \\ I_3 &= \frac{h_+}{h_+ + h_-} \frac{u_-}{u}, & I_4 &= \frac{h_-}{uu_-}. \end{aligned} \tag{5.83}$$

Any three of these are independent; the four satisfy the identity

$$I_1 I_2 = I_3 I_4. \tag{5.84}$$

An invariant difference scheme can be written as

$$2(I_2 + I_3 - 1) = I_1^2 I_2 \frac{I_2 + I_3}{I_3} f(I_1 I_2), \qquad I_1 + I_4 = 4 I_1 I_2 g(I_1 I_2), \tag{5.85}$$

i.e.

$$
\begin{aligned}
u_{x\bar{x}} &= \frac{1}{h_+ + h_-}\frac{1}{u^2}\left(\frac{h_+}{u_+} + \frac{h_-}{u_-}\right) f\left(\frac{1}{u^2}\frac{h_+h_-}{h_+ + h_-}\right), \\
\frac{h_+}{u_+} + \frac{h_-}{u_-} &= \frac{4}{u}\frac{h_+h_-}{h_+ + h_-} g\left(\frac{1}{u^2}\frac{h_+h_-}{h_+ + h_-}\right).
\end{aligned}
\tag{5.86}
$$

For $f = g = 1$ this scheme approximates the ODE (5.81). We see that this $sl(2, \mathcal{C})$ invariant OΔS is quite different from the standard discretization (5.20).

$A_{3,4}$

$$
\hat{X}_1 = \partial_u, \qquad \hat{X}_2 = x\partial_x + u\partial_u, \qquad \hat{X}_3 = x^2\partial_x + (-x^2 + u^2)\partial_u. \tag{5.87}
$$

This algebra is again $sl(2, \mathbb{C})$ and can be transformed into

$$
\hat{Y}_1 = \partial_x + \partial_u, \qquad \hat{Y}_2 = x\partial_x + u\partial_u, \qquad \hat{Y}_3 = x^2\partial_x + u^2\partial_u. \tag{5.88}
$$

The realization (5.88) (and hence also (5.87)) is imprimitive; (5.76) is primitive. Hence $A_{3,4}$ and $A_{3,3}$ are not equivalent. The invariant ODE for the algebra (5.87) is

$$
x\ddot{u} = C(1 + \dot{u}^2)^{\frac{3}{2}} + \dot{u}(1 + \dot{u}^2), \tag{5.89}
$$

where C is a constant. The general integral of equation (5.89) can be written as

$$
(x - x_0)^2 + (u - u_0)^2 = \left(\frac{x_0}{C}\right)^2, \qquad C \neq 0, \tag{5.90}
$$

$$
x^2 + (u - u_0)^2 = x_0^2, \qquad C = 0, \tag{5.91}
$$

where x_0 and u_0 are integration constants.

The difference invariants corresponding to the algebra (5.88) are

$$
\begin{aligned}
I_1 &= \frac{x_+ - x}{x_+ x}\left(1 + u_x^2\right), \qquad I_2 = \frac{x - x_-}{x_- x}\left(1 + u_{\underline{x}}^2\right), \\
I_3 &= -\frac{(x_+ - x)(x - x_-)}{2xx_+x_-}\left\{(h_+u_x^2 + x_+ + x)u_{\underline{x}} + (h_-u_{\underline{x}}^2 - x_- - x)u_x\right\}.
\end{aligned}
\tag{5.92}
$$

An invariant scheme representing the ODE (5.89) can be written as

$$
I_3 = C\left(\frac{I_1 + I_2}{2}\right)^{\frac{3}{2}}, \qquad I_1 = I_2, \tag{5.93}
$$

(this is not the most general such scheme).

5.3.4. Lagrangian formalism and solutions of three-point OΔS. In section 5.3.2 we presented a Lagrangian formalism for second-order ODE's. Let us now adapt it to OΔS [67].

The Lagrangian density (5.48) will now be a two-point function

$$
\mathcal{L} = \mathcal{L}(x, u, x_+, u_+). \tag{5.94}
$$

Instead of the Euler–Lagrange equation (5.49) we have two *quasiextremal equations* [60, 64, 67] corresponding to 'discrete variational derivatives' of $\mathcal{L}$ with respect to x and y independently

$$
\frac{\delta\mathcal{L}}{\delta x} = h_+\frac{\partial\mathcal{L}}{\partial x} + h_-\frac{\partial\mathcal{L}^-}{\partial x} + \mathcal{L}^- - \mathcal{L} = 0, \qquad \frac{\delta\mathcal{L}}{\delta u} = h_+\frac{\partial\mathcal{L}}{\partial u} + h_-\frac{\partial\mathcal{L}^-}{\partial u} = 0 \tag{5.95}
$$

where $\mathcal{L}^-$ is obtained by downshifting $\mathcal{L}$ (replacing n by $n-1$ everywhere). In the continuous limit both quasiextremal equations reduce to the same Euler–Lagrange equation. Thus, the two quasiextremal equations together can be viewed as an OΔS, where e.g. the difference between them defines the lattice.

The Lagrangian density (5.94) will be divergence invariant under the transformation generated by vector field $\hat{X}$, if it satisfies

$$\mathrm{pr}\,\hat{X}(\mathcal{L}) + \mathcal{L}D_+(\xi) = D_+(V) \tag{5.96}$$

for some function $V(x, u)$ where $D_+(f)$ is the discrete total derivative

$$D_+f(x,u) = \frac{f(x+h, u(x+h)) - f(x,u)}{h} \tag{5.97}$$

Each infinitesimal Lagrangian divergence symmetry operator X will provide one first integral of the quasiextremal equation

$$h_-\phi\frac{\partial\mathcal{L}^-}{\partial u} + h_-\xi\frac{\partial\mathcal{L}^-}{\partial x} + \xi\mathcal{L}^- - V = K \tag{5.98}$$

[67]. These first integrals will have the form

$$f_a(x, x_+, u, u_+) = K_a, \qquad a = 1, \ldots. \tag{5.99}$$

Thus, if we have two first integrals, we are left with a two point OΔS to solve. If we have three first integrals, then the quasiextremal equations reduce to a single two-point difference equation, e.g. involving just x_n and x_{n+1}. This can often be solved explicitly [196].

This procedure has been systematically applied to three-point OΔS in the original article [67]. For brevity we will just consider some examples here.

Let us first consider a two-dimensional Abelian Lie algebra and the corresponding invariant second-order ODE:

$$\hat{X}_1 = \partial_x, \qquad \hat{X}_2 = \partial_u, \qquad \ddot{u} = F(\dot{u}). \tag{5.100}$$

This equation is the Euler–Lagrange equation for the Lagrangian

$$\mathcal{L} = u + G(\dot{u}), \qquad \ddot{G} = \frac{1}{F}, \tag{5.101}$$

and both symmetries are Lagrangian ones

$$\mathrm{pr}\,\hat{X}_1\mathcal{L} + \mathcal{L}D(\xi_1) = 0, \qquad \mathrm{pr}\,\hat{X}_2\mathcal{L} + \mathcal{L}D(\xi_2) = 1 = D(x). \tag{5.102}$$

The corresponding two first integrals are

$$J_1 = u + G(\dot{u}) - \dot{u}\dot{G}(\dot{u}), \qquad J_2 = \dot{G}(\dot{u}) - x. \tag{5.103}$$

Introducing H as the inverse function of $\dot{G}$ we have

$$\dot{u} = H(J_2 + x), \qquad H(J_2 + x) = [\dot{G}]^{-1}(J_2 + x). \tag{5.104}$$

Substituting into the first equation in equation (5.103), we obtain the general solution of (5.100) as

$$u(x) = J_1 - G[H(J_2 + x)] + (J_2 + x)H(J_2 + x). \tag{5.105}$$

Now let us consider the discrete case. We introduce the discrete analogue of (5.101) as

$$\mathcal{L} = \frac{u + u_+}{2} + G(u_x) \tag{5.106}$$

for some smooth function G. Equations (5.102) hold (with D interpreted as the discrete total derivative D_+). The two quasiextremal equations are

$$\begin{aligned} &\frac{x_+ - x_-}{2} - \dot{G}(u_x) + \dot{G}(u_{\underline{x}}) = 0, \\ &u_x\dot{G}(u_x) - u_{\underline{x}}\dot{G}(u_{\underline{x}}) - G(u_x) + G(u_{\underline{x}}) - \frac{u_+ - u_-}{2} = 0. \end{aligned} \tag{5.107}$$

The two first integrals obtained using Noether's theorem in this case can be written as

$$\dot{G}(u_x) - \frac{x + x_+}{2} = B \tag{5.108}$$

$$-u_x\dot{G}(u_x) + G(u_x) + u + \frac{1}{2}(x_+ - x)u_x = A. \tag{5.109}$$

In principle, these two integrals can be solved to obtain

$$u_x = H\left[B + \tfrac{1}{2}(x_+ + x)\right], \qquad u = \Phi(A, B, x, x_+), \tag{5.110}$$

where $H[z] = [\dot{G}]^{-1}(z)$ and Φ is obtained by solving equation (5.109), once $u_x = H$ is substituted into this equation. A three-point difference equation for x_{n+2}, x_{n+1} and x_n, not involving u is obtained from the consistency condition $u_x = \frac{u_{n+1}-u_n}{x_{n+1}-x_n}$. In general this equation is difficult to solve. We shall follow a different procedure which is less general, but works well when the considered OΔS has a three-dimensional solvable symmetry algebra with $\{\partial_x, \partial_u\}$ as a subalgebra. We add a third equation to the system (5.108), (5.109), namely

$$\frac{x_+ - x}{x - x_-} = 1 + \epsilon. \tag{5.111}$$

The general solution of equation (5.111) is

$$x_n = (x_0 + B)(1 + \epsilon)^n - B \tag{5.112}$$

where x_0 and B are integration constants. We will identify B with the constant in equation (5.108), but leave ϵ as an arbitrary constant. Equation (5.112) defines an exponential lattice (for $\epsilon \neq 0$). Using (5.112) together with (5.108) and (5.109), we find

$$u_x = H\left[(x_n + B)\left(1 + \frac{\epsilon}{2}\right)\right], \qquad H[z] = [\dot{G}]^{-1}(z) \tag{5.113}$$

$$u_n = A + (x_n + B)H\left[(x_n + B)\left(1 + \frac{\epsilon}{2}\right)\right] - G\left[H(x_n + B)\left(1 + \frac{\epsilon}{2}\right)\right]. \tag{5.114}$$

There is no guarantee that equations (5.113) and (5.114) are compatible. However, let us consider the two special cases with three-dimensional solvable symmetry algebras, namely algebras $A_{3,1}$ and $A_{3,2}$ of section 5.3.3.

Algebra $A_{3,1}$. We choose $G(u_x)$ to be

$$G(u_x) = \frac{(k-1)^2}{k}u_x^{\frac{k}{k-1}}, \qquad k \neq 0, 1. \tag{5.115}$$

From equations (5.113) and (5.114) we obtain

$$u_x = \left(\frac{1}{k-1}\right)^{k-1} x_n^{k-1}\left(1 + \frac{\epsilon}{2}\right)^{k-1} \tag{5.116}$$

$$u_n = \frac{1}{k}\left(\frac{1}{k-1}\right)^{k-1}(x_n + B)^k\left(1 + \frac{\epsilon}{2}\right)^{k-1}\left[1 + (1-k)\frac{\epsilon}{2}\right]. \tag{5.117}$$

The consistency condition (for u_x to be the discrete derivative of u_n) provides us with a transcedental equation for ϵ:

$$[(1+\epsilon)^k - 1]\left[1 + (1-k)\frac{\epsilon}{2}\right] = k\epsilon. \tag{5.118}$$

In the continuous limit we take $\epsilon \to 0$ and u_n given by equation (5.117) goes to the general solution of the ODE (5.71). In equation (5.118) terms of order ϵ^0, ϵ and ϵ^2 cancel. The solution u_n coincides with the continuous limit up to terms of order ϵ^2.

We mention that in the special case $k = -1$ all three symmetries of the OΔS are Lagrangian ones and in this case equation (5.118) is identically satisfied for any ϵ.

Algebra $A_{3,2}$. We choose $G(u_x)$ to be

$$G(u_x) = \mathrm{e}^{u_x} \tag{5.119}$$

and obtain

$$u_x = \ln(x_n + B)\left(1 + \frac{\epsilon}{2}\right), \tag{5.120}$$

$$u_n = (x_n + B)\ln(x_n + B) + A + (x_n + B)\left[\ln\left(1 + \frac{\epsilon}{2}\right) - \left(1 + \frac{\epsilon}{2}\right)\right]. \tag{5.121}$$

Expressions (5.120) and (5.121) are consistent if ϵ satisfies

$$\epsilon\left(1 + \frac{\epsilon}{2}\right) - (1+\epsilon)\ln(1+\epsilon) = 0. \tag{5.122}$$

Again (5.121) coincides with its continuous limit up to terms of order ϵ^2 and in (5.122) terms of order ϵ^0, ϵ^1 and ϵ^2 cancel.

For the $sl(2,\mathbb{R})$ algebra $A_{3,3}$ all three symmetry operators $\hat{X}_1$, $\hat{X}_2$ and $\hat{X}_3$ correspond to Lagrangian symmetries. The corresponding OΔS is integrated in [67].

5.4. Symmetries of partial differential schemes

As in the introduction we will restrict to PΔS with one dependent variable u and two independent ones x and t.

Point transformations will be generated by vector fields of the form (1.25) and their prolongation must act on all points of the lattice figuring in equation (1.18) and is given by equation (1.26).

The continuous transformations generated by the vector field (1.25) will leave the solution set of the system (1.18) invariant if we have

$$\mathrm{pr}\,\hat{X}E_a|_{E_b=0} = 0, \qquad a = 1,\ldots,n_E; \quad b = 1,\ldots,n_E. \tag{5.123}$$

Equation (5.123) provides a system of determining equations for the coefficients ξ, τ and ϕ in equation (1.25), just as in the case of OΔS.

Most of the literature on symmetries of PΔS is devoted to the invariant discretization of partial differential equations [15, 32, 62, 65, 68]. An exception is [160] where the form of the PΔS was postulated and its symmetry group was calculated.

Here we shall give a new example of the second procedure.

Let us consider the continuous wave equation

$$u_{xt} = 0 \tag{5.124}$$

and its discretization (1.23) given in the introduction. The PDE (5.124) is conformally invariant and linear. Its symmetry algebra is infinite dimensional and is realized by the vector fields

$$\begin{aligned} &\hat{T}(f) = f(t)\partial_t, \qquad \hat{X}(h) = h(x)\partial_x,\\ &\hat{L} = u\partial_u, \qquad \hat{S}_1(k) = k(t)\partial_u, \qquad \hat{S}_2(l) = l(x)\partial_u, \end{aligned} \tag{5.125}$$

where $f(t), h(x), k(t)$ and $l(x)$ are arbitrary functions of the indicated variables. $\hat{T}(f)$ and $\hat{X}(h)$ correspond to conformal transformations, in this case an arbitrary invertible reparametrization of t and x (separately). $\hat{S}_1(k)$ and $\hat{S}_2(l)$ simply represent the linear superposition principle: we can add an arbitrary solution of equation (5.124) to any given solution. The vector field $\hat{L}$, present for any linear equation, only tells us that the constant multiple of a solution is also a solution.

Now let us consider the system (1.23) of the introduction. Applying $\mathrm{pr}\,\hat{X}$ of equation (1.26) to E_2 and E_3 we find $\xi = \xi(x), \tau = \tau(t)$. On the lattice satisfying $E_2 = E_3 = 0$ equation $E_1 = 0$ can be rewritten as

$$u(x_{m,n+1}, t_{m+1,n}) - u(x_{m,n}, t_{m+1,n}) - u(x_{m,n+1}, t_{m,n}) + u(x_{m,n}, t_{m,n}) = 0. \tag{5.126}$$

Applying $\mathrm{pr}\,\hat{X}$ to (5.126) we find $\phi = \phi_1(x) + \phi_2(t) + cu$, where $\xi(x)$, $\tau(t)$, $\phi_1(x)$ and $\phi_2(t)$ are arbitrary functions and c is an arbitrary constant. Thus, the symmetry algebra of the PΔS (1.23) is given by equation (5.125) and coincides with that of its continuous limit. Conformal invariance of the PΔS in this case means that we have an orthogonal lattice but the spacings on each axis are unspecified. This also explains why the solutions of the scheme (1.23), given by equation (1.24) coincide with those of equation (5.124) for any choice of $\alpha(m)$ and $\beta(n)$ in equation (1.24).

If we impose any further conditions on the lattice we will reduce the symmetry group. For instance, if we request that the spacing be regular by imposing

$$E_4 = t_{m+1,n} - 2t_{m,n} + t_{m-1,n} = 0, \qquad E_5 = x_{m,n+1} - 2x_{m,n} + x_{m,n-1} = 0, \tag{5.127}$$

we loose conformal invariance and keep only translational and dilational invariance:

$$\tau = \tau_1 t_{m,n} + \tau_0, \qquad \xi = \xi_1 x_{m,n} + \xi_0, \tag{5.128}$$

where τ_i and ξ_i are constants.

We wish to stress that the PDE (5.124) is exceptional in having such a simple invariant discretization. The known invariant discrete models for the heat equation, the Korteweg–de Vries equation or the nonlinear Schrödinger equation are much more involved [15, 32, 62, 65, 68, 265].

6. Conclusions and open problems

Let us compare the symmetry approach for difference equations with that for differential ones. In both cases one is interested in transformations taking solutions into solutions and in both cases, choices have to be made. Lie's choice of point symmetries of differential equations is so natural that it is often forgotten that it is also just an Ansatz: the vector fields should depend on the independent and dependent variables only. Generally speaking, this Ansatz has turned out to be the most useful one. Contact symmetries have much fewer applications than point ones. Generalized symmetries are mainly of use for identifying integrable nonlinear partial differential equations.

In this review we have stressed the fact that for difference equations this choice of pure point transformations must be modified. Without significant modifications it remains mainly fruitful for differential–difference equations rather than for purely difference ones (see section 2).

Which modifications are needed for difference equations depends on the application that we have in mind. For differential equations there are two main types of applications in physics. In the first, the equations are already known and group theory is used to solve them. In the second, the symmetries of the physical problem are known and are used in building up the theoretical model, i.e. the symmetries precede the equations. These two aspects are also present in the case of difference equations, but there are new features. First of all, the physical processes that are being described may be discrete and the lattices involved may be real physical objects. If we are considering linear theories, like quantum mechanics, or quantum field theory on a lattice, then the generalized point symmetries of section 3 are extremely promising. A mathematical tool, umbral calculus, is ready to be used, both to solve equations and also to formulate models. For nonlinear theories on given fixed lattices the most appropriate symmetry approach involves generalized symmetries, as reviewed in section 4. Their main application is as in the continuous case: to identify integrable systems on lattices. Moreover it can be used to get new interesting solutions. An interesting feature is that some point symmetries of differential equations, in particular dilations, appear as generalized symmetries for difference equations.

The second type of application of difference equations in physics is more practical and in a certain sense, less fundamental. We have in mind the situation when the physical processes are really continuous and are described by differential equations. These are then discretized in order to solve them. The lattices used are then our choice and they can be chosen in a symmetry adapted way. Moreover, as shown in section 5, the difference equation and lattice are both part of a 'difference scheme' and the actual lattice is part of the solution of this scheme. We can then restrict ourselves to point transformations, but they act simultaneously on the solutions and on the lattice.

In an attempt to keep this review reasonably short, we have left out many interesting and important topics. Among them we have not included a complete discussion of partial difference equations on transforming symmetry-adapted lattices [65, 68, 160, 260] and the use of Lie point symmetries to get conditions for the linearizability of difference equations [33, 98, 137, 187, 223, 236, 238]. Also left out is the vast area of numerical methods of solving differential equations, making use of their symmetry properties [266] or the treatment of asymptotic symmetries for difference equations [89]. Very active areas of research, not covered in the present review, are the use of higher symmetries to identify integrable lattice equations [10, 11, 175–178, 245, 268], and also discrete versions of the Painlevé text [2, 52, 53, 95–97, 108–111, 133, 210, 224–226, 251, 256].

Since we are reviewing a relatively new area of research, many open questions remain. Thus, for purely difference equations on fixed lattices, the role of discrete symmetries has not been fully explored. Basically, what is needed, specially for partial difference equations, is to apply results from crystallography to characterize discrete or finite transformations taking a lattice into itself. For differential–difference equations with three independent variables, one or two of them discrete, a classification of equations with Kac–Moody–Virasoro symmetries should help in identifying new integrable lattice equations. Applications to genuine physical systems would be of great interest. The umbral approach of section 3 has so far been applied in a rather formal manner. The question of the convergence of formal power series solutions must be addressed. The main question is whether one can develop a complete convergent quantum theory on a lattice. The generalized symmetries of section 4 are an integral part of the theory of integrability on a lattice. There the greatest challenge lies in the field of applications, i.e in applying the techniques of integrability to the real world of discrete phenomena. Finally, the greatest challenge in the direction of symmetry adapted discretizations is to establish whether they provide improved numerical methods, in particular for partial differential equations, or higher order ordinary ones.

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References

[1] Ablowitz M J and Clarkson P A 1991 *Solitons, Nonlinear Evolution Equations and Inverse Scattering* (Cambridge: Cambridge University Press)

[2] Ablowitz M J, Halburd R and Herbst B 2000 On the extension of the Painlevé property to difference equations *Nonlinearity* **13** 889–905

[3] Ablowitz M J, Kaup D J, Newell A C and Segur H 1974 The inverse scattering transform—Fourier analysis for nonlinear problems *Stud. Appl. Math.* **53** 249–318

[4] Ablowitz M J and Ladik J F 1975 Nonlinear differential–difference equations *J. Math. Phys.* **16** 598–603

[5] Ablowitz M J and Ladik J F 1976 A nonlinear difference scheme and inverse scattering *Stud. Appl. Math.* **55** 213–29

[6] Ablowitz M J and Segur H 1981 *Solitons and the Inverse Scattering Transform* (Philadelphia: SIAM)

[7] Aczel J 1966 *Lectures on Functional Equations and Their Applications* (New York: Academic)

[8] Aczel J (ed) 1984 *Functional Equations: History, Applications and Theory* (Amsterdam: Reidel)

[9] Adler V E 2001 Discrete equations on planar graphs *J. Phys. A: Math. Gen.* **34** 10453–60

[10] Adler V E, Shabat A B and Yamilov R 2000 Symmetry approach to the integrability problem *Theor. Math. Phys.* **125** 1603–61

[11] Adler V E, Svinolupov S I and Yamilov R I 1999 Multi-component Volterra and Toda type integrable equations *Phys. Lett.* A **254** 24–36

[12] Ames W F 1972 *Nonlinear Partial Differential Equations in Engineering* (New York: Academic)

[13] Anderson R L and Ibragimov N H 1979 *Lie–Bäcklund Transformations in Applications* (Philadelphia: SIAM)

[14] Baĭkov V A, Gazizov R K and Ibragimov N Kh 1988 Approximate symmetries *(Russian) Mat. Sb. (N.S.)* **136** 435–50
Baĭkov V A, Gazizov R K and Ibragimov N Kh 1988 *(Russian) Mat. Sb. (N.S.)* **178** 590
Baĭkov V A, Gazizov R K and Ibragimov N Kh 1989 *Math. USSR-Sb.* **64** 427–41 (Engl. Transl.)

[15] Bakirova M, Dorodnitsyn V and Kozlov R 1997 Symmetry-preserving difference schemes for heat transfer equations with a source *J. Phys. A: Math. Gen.* **30** 8139–55

[16] Baumann G 2000 *Symmetry Analysis of Differential Equations with Mathematica* (New York: Springer)

[17] Bellon M P and Viallet C M 1999 Algebraic entropy *Commun. Math. Phys.* **204** 425–37

[18] Bessel-Hagen E 1921 Uber die Erhaltungssatze der Elektrodynamik *Math. Ann.* **84** 258–76

[19] Bluman G W 1990 Simplifying the form of Lie groups admitted by a given differential equation *J. Math. Anal. Appl.* **145** 52–62

[20] Bluman G and Anco S 2002 *Symmetry and Integration Methods for Differential Equations* (New York: Springer)

[21] Bluman G W and Cole J D 1969 The general similarity solutions of the heat equation *J. Math. Mech.* **18** 1025–42

[22] Bluman G W and Kumei S 1989 *Symmetries of Differential Equations* (New York: Springer)

[23] Boiti M, Bruschi M, Pempinelli F and Prinari B 2003 A discrete Schrödinger spectral problem and associated evolution equations *J. Phys. A: Math. Gen.* **36** 139–49

[24] Boutin M 2002 On orbit dimensions under a simultaneous Lie group action on n copies of a manifold *J. Lie Theory* **12** 191–203

[25] Bruschi M, Levi D and Ragnisco O 1980 Extension of the Zakharov–Shabat generalized inverse method to solve differential–difference and difference–difference equations *Nuovo Cimento* A **58** 56–66

[26] Bruschi M, Levi D and Ragnisco O 1980 Toda lattice and generalised Wronskian technique *J. Phys. A: Math. Gen.* **13** 2531–3

[27] Bruschi M, Manakov S V, Ragnisco O and Levi D 1980 The nonabelian Toda lattice—discrete analogue of the matrix Schrödinger equation *J. Math. Phys.* **21** 2749–53

[28] Bruschi M and Ragnisco O 1980 Existence of a Lax pair for any member of the class of nonlinear evolution equations associated to the matrix Schrödinger spectral problem *Lett. Nuovo Cimento* **29** 321–6

[29] Bruschi M and Ragnisco O 1981 Nonlinear differential–difference matrix equations with n-dependent coefficients *Lett. Nuovo Cimento* **31** 492–6

[30] Bruschi M and Ragnisco O 1981 Nonlinear differential–difference equations, associated Bäcklund transformations and Lax technique *J. Phys. A: Math. Gen.* **14** 1075–81

[31] Bruschi M, Ragnisco O and Levi D 1981 Evolution equations associated with the discrete analog of the matrix Schrödinger spectral problem solvable by the inverse spectral transform *J. Math. Phys.* **22** 2463–71

[32] Budd C and Dorodnitsyn V A 2001 Symmetry-adapted moving mesh schemes for the nonlinear Schrödinger equation *J. Phys. A: Math. Gen.* **34** 10387–400

[33] Byrnes G B, Sahadevan R and Quispel G R W 1995 Factorizable Lie symmetries and the linearization of difference equations *Nonlinearity* **8** 443–59

[34] Calogero F 1975 Bäcklund transformations and functional relation for solutions of nonlinear partial differential equations solvable via the inverse-scattering method *Lett. Nuovo Cimento* **14** 537–43

[35] Calogero F and Degasperis A 1982 *Spectral Transform and Solitons: Tools to Solve and Investigate Nonlinear Evolution Equations* (Amsterdam: North Holland)

[36] Campbell D K, Rosenau P and Zaslavsky G M (ed) 2005 Focus issue: the 'Fermi–Pasta–Ulam' problem—the first 50 years *Chaos* **15** articles 015101–015121

[37] Cantwell B J 2002 *Introduction to Symmetry Analysis* (Cambridge: Cambridge University Press)

[38] Case K M 1973 On discrete inverse scattering problems. II. *J. Math. Phys.* **14** 916–20

[39] Case K M and Chiu S C 1973 The discrete version of the Marchenko equations in the inverse scattering problem *J. Math. Phys.* **14** 1643–7

[40] Case K M and Kac M 1973 A discrete version of the inverse scattering problem *J. Math. Phys.* **14** 594–603

[41] Celledoni E, Iserles A, Nørsett S P and Orel B 2002 Complexity theory for Lie-group solvers *J. Complexity* **18** 242–86

[42] Champagne B, Hereman W and Winternitz P 1991 The computer calculation of Lie point symmetries of large systems of differential equations *Comput. Phys. Commun.* **66** 319–40

[43] Champagne B and Winternitz P 1988 On the infinite dimensional symmetry group of the Davey–Stewartson equation *J. Math. Phys.* **29** 1–8

[44] Chen W Y C, Galbraith H W and Louck J D 2001 Angular momentum theory, umbral calculus, and combinatorics *Comp. Math. Appl.* **41** 1199–214

[45] Chiu S C and Ladik J F 1977 Generating exactly soluble nonlinear discrete evolution equations by a generalized Wronskian technique *J. Math. Phys.* **18** 690–700

[46] Chu M T 1984 The generalized Toda flow, the QR algorithm and the center manyfold theory *SIAM J. Alg. Disc. Meth.* **5** 187–201

[47] Cicogna G 2004 Weak symmetries and adapted variables for differential equations *Int. J. Geom. Meth. Mod. Phys.* **1** 23–31

[48] Cicogna G and Gaeta G 1999 *Symmetry and Perturbation Theory in Nonlinear Dynamics* (Berlin: Springer)

[49] Cicogna G and Gaeta G 2001 Partial Lie-point symmetries of differential equations *J. Phys. A: Math. Gen.* **34** 491–512

[50] Clarkson P A and Kruskal M D 1989 New similarity reductions of the Boussinesq equation *J. Math. Phys.* **30** 2201–13

[51] Clarkson P A and Nijhoff F W (ed) 1999 *Symmetries and Integrability of Difference Equations (LMS Lecture Note Series vol 255)* (Cambridge: Cambridge University Press)

[52] Conte R and Musette M 1996 A new method to test discrete Painlevé equations *Phys. Lett.* A **223** 439–48

[53] Costin O and Kruskal M 2002 Movable singular points of solutions of difference equations, their relation to solvability, and the investigation of superstable fixed points *Theor. Math. Phys.* **133** 1455–62

[54] David D, Kamran N, Levi D and Winternitz P 1986 Subalgebras of loop algebras and symmetries of the Kadomtsev–Petviashvili equation *Phys. Rev. Lett.* **55** 2111–3

[55] David D, Kamran N, Levi D and Winternitz P 1986 Symmetry reduction of the Kadomtsev–Petviashvili equation using a loop algebra *J. Math. Phys.* **27** 1225–37

[56] DiBucchianico A and Loeb D 2000 Umbral Calculus *Electr. J. Comb.* D **S3** http://www.combinatorics.org

[57] Dimakis A, F Müller-Hoissen F and Striker T 1996 Umbral calculus, discretization, and quantum mechanics on a lattice *J. Phys. A: Math. Gen.* **29** 6861–76

[58] Dodd R K 1978 Generalised Backlund transformation for some non-linear partial differential–difference equations *J. Phys. A: Math. Gen.* **11** 81–92

[59] Dorodnitsyn V A 1991 Transformation groups in net spaces *J. Math. Sciences* **55** 1490–517

[60] Dorodnitsyn V A 1993 Finite difference analog of the Noether theorem *Dokl. Ak. Nauk* **328** 678–82

[61] Dorodnitsyn V A 1994 Finite difference models entirely inheriting continuous symmetries of differential equations *Int. J. Mod. Phys.* C **5** 723–34

[62] Dorodnitsyn V A 1994 Invariant discrete model for the Korteweg–de Vries equation *Preprint* CRM-2187, Université de Montréal

[63] Dorodnitsyn V A 1993 Symmetry of finite difference equations, Chapter 17 of *CRC Handbook of Lie Group Analysis of Differential Equations. Vol. I: Symmetries, Exact Solutions and Conservation Laws* ed N H Ibragimov (Baton Rouge: CRC Press)

[64] Dorodnitsyn V A 2001 *Group Properties of Difference Equations* (Moscow: Fizmatlit) (in Russian)

[65] Dorodnitsyn V A and Kozlov R 2003 A heat transfer with a source: the complete set of invariant difference schemes *J. Nonlinear Math. Phys.* **10** 16–50

[66] Dorodnitsyn V A, Kozlov R and Winternitz P 2000 Lie group classification of second order difference equations *J. Math. Phys.* **41** 480–509

[67] Dorodnitsyn V A, Kozlov R and Winternitz P 2004 Continuous symmetries of Lagrangians and exact solutions of discrete equations *J. Math. Phys.* **45** 336–59

[68] Dorodnitsyn V A and Winternitz P 2000 Lie point symmetry preserving discretizations for variable coefficient Korteweg–de Vries equations *Nonlinear Dyn.* **22** 49–59

[69] Durán A 1999 *Symmetries of Differential Equations and Numerical Applications* (Coimbra: Universidade de Coimbra)

[70] Elaydi S 2005 *An Introduction to Difference Equations* (New York: Springer)

[71] Faddeev L D and Takhtajan L A 1987 *Hamiltonian Methods in the Theory of Solitons* (Berlin: Springer)

[72] Flaschka H 1974 The Toda lattice: I. Existence of integrals *Phys. Rev.* B **9** 1924–5

[73] Flaschka H 1974 On the Toda lattice: II. Inverse scattering solutions *Prog. Theor. Phys.* **51** 703–16

[74] Fermi E, Pasta J and Ulam S 1955 Los Alamos Rpt. LA1940 included in 1965 *Fermi, Enrico Collected Papers (Note e memorie). Vol. II: United States, 1939–1954* (Chicago, IL: The University of Chicago Press; Rome: Accademia Nazionale dei Lincei)

[75] Floreanini R, Negro J, Nieto L M and Vinet L 1996 Symmetries of the heat equation on a lattice *Lett. Math. Phys.* **36** 351–5

[76] Floreanini R and Vinet L 1995 Lie symmetries of finite difference equations *J. Math. Phys.* **36** 7024–42

[77] Fokas A S 1980 A symmetry approach to exactly solvable evolution equations *J. Math. Phys.* **21** 1318–25

[78] Ford J 1992 The Fermi–Pasta–Ulam problem: paradox turns discovery *Phys. Rep.* **213** 271–310

[79] Fordy A P and Gibbons J 1980 Integrable nonlinear Klein–Gordon equations and Toda lattices *Commun. Math. Phys.* **77** 21–30

[80] Fushchich V I, Serov N I and Chopik V I 1988 Conditional invariance and nonlinear heat equations (Russian, English summary) *Dokl. Akad. Nauk Ukrain. SSR Ser.* A **86** 17–21

[81] Fushchich V I, Shtelen W M and Slavutsky S L 1991 Reduction and exact solutions of the Navier–Stokes equations W.I. *J. Phys. A: Math. Gen.* **24** 971–84

[82] Fushchich W I and Shtelen W M 1989 On approximate symmetry and approximate solutions of the non-linear wave equation with a small parameter *J. Phys. A: Math. Gen.* **22** L887–90

[83] Fuchssteiner B 1979 Application of hereditary symmetries to nonlinear evolution equations *Nonlinear Anal. Theory Math. Appl.* **3** 849–62

[84] Fuchssteiner B 1983 Mastersymmetries, high order time-dependent symmetries and conserved densities of nonlinear evolution equations *Prog. Theor. Phys.* **70** 1508–22

[85] Fuchssteiner B and Fokas A S 1981/1982 Symplectic structures, their Bäcklund transformations and hereditary symmetries *Physica* D **4** 47–66

[86] Fuchssteiner B 1981 The Lie algebra structure of nonlinear evolution equations admitting infinite-dimensional Abelian symmetry groups *Prog. Theor. Phys.* **65** 861–76

[87] Gaeta G 1994 *Nonlinear Symmetries and Nonlinear Equations* (Dordrecht: Kluwer)

[88] Gaeta G 1994 Asymptotic symmetries and asymptotically symmetric solutions of partial differential equations *J. Phys. A: Math. Gen.* **27** 437–51

[89] Gaeta G, Levi D and Mancinelli R 2005 Asymptotic symmetries of difference equations *J. Nonlinear Math. Phys.* **12** Suppl. 2

[90] Gaeta G and Mancinelli R 2005 Asymptotic scaling in a model class of anomalous reaction-diffusion equations *J. Nonlinear Math. Phys.* **12** 550–66

[91] Gazeau J P and Winternitz P 1992 Symmetries of variable coefficient Korteweg–de Vries equation *J. Math. Phys.* **33** 4087–102

[92] Gomez-Ullate D, Lafortune S and Winternitz P 1999 Symmetries of discrete dynamical systems involving two species *J. Math. Phys.* **40** 2782–804

[93] Gonzalez-Lopez A, Kamran N and Olver P J 1992 Lie algebras of vector fields in the real plane *Proc. London Math. Soc.* **64** 339–68

[94] Gorskii A Z and Szmigielski 1998 On pairs of difference operators satisfying $[D, x] = Id$ *J. Math. Phys.* **39** 545–68

[95] Grammaticos B and Ramani R 2004 Discrete Painlevé equations: a review *Discrete Integrable Systems* ed B Grammaticos, Y Kosmann-Schwarzbach and T Tamizhmani (Berlin: Springer) pp 245–321

[96] Grammaticos G, Ramani A and Papageorgiou V 1991 Do integrable mappings have the Painlevé property *Phys. Rev. Lett.* **67** 1825–8

[97] Grammaticos B, Tamizhmani T, Ramani A and Tamizhmani K M 2001 Growth and integrability in discrete systems *J. Phys. A: Math. Gen.* **34** 3811–21

[98] Haggar F A, Byrnes G B, Quispel G R W and Capel H W 1996 K-integrals and k-Lie symmetries in discrete dynamical systems *Physica* A **233** 379–94

[99] Helgason S 1962 *Differential Geometry and Symmetric Spaces* (New York: Academic)

[100] Hereman W 1996 Symbolic software for Lie symmetry analysis *Lie Group Analysis of Differential Equations* vol 3, ed N H Ibragimov (Boca Raton, FL: CRC Press) pp 367–413
[101] Hernández Heredero R and Levi D 2003 The discrete nonlinear Schrödinger equation and its Lie symmetry reductions *J. Nonlinear Math. Phys.* **10** (Suppl. 2) 77–94
[102] Hernández Heredero R, Levi D, Rodríguez M A and Winternitz P 2000 Lie algebra contractions and symmetries of the Toda hierarchy *J. Phys. A: Math. Gen.* **33** 5025–40
[103] Hernández Heredero R, Levi D, Rodríguez M A and Winternitz P 2001 Relation between Bäcklund transformations and higher continuous symmetries of the Toda equation *J. Phys. A: Math. Gen.* **34** 2459–65
[104] Hernández Heredero R, Levi D, Rodríguez M A and Winternitz P 2001 Symmetries of differential difference equations and Lie algebra contractions *Bäcklund & Darboux Transformations: The Geometry of Soliton Theory* ed A Coley, D Levi, R Milson, C Rogers and P Winternitz (Providence, RI: American Mathematical Society)
[105] Hernández Heredero R, Levi D and Winternitz P 1999 Symmetries of the discrete Burgers equation *J. Phys. A: Math. Gen.* A **32** 2685–95
[106] Hernández Heredero R, Levi D and Winternitz P 2000 Symmetry preserving discretization of the Burgers equation *Symmetries and Integrability of Difference Equations* ed D Levi and O Ragnisco (Providence, RI: American Mathematical Society) pp 197–208
[107] Hernández Heredero R, Levi D and Winternitz P 2001 Symmetries of the discrete nonlinear Schrödinger equation *Theor. Math. Phys.* **127** 729–37
[108] Hietarinta J 2001 Integrability tests for difference equations *Nonlinear Anal.* **47** 4641–50
[109] Hietarinta J 2004 A new two-dimensional lattice model that is 'consistent around a cube' *J. Phys. A: Math. Gen.* **37** L67–73
[110] Hietarinta J and Viallet C 1998 Singularity confinement and chaos in discrete systems *Phys. Rev. Lett.* **81** 325–8
[111] Hietarinta J and Viallet C 2000 Discrete Painlevé I and singularity confinement in projective space *Chaos Solitons Fractals* **11** 29–32
[112] Hill J M 1992 *Differential Equations and Group Methods for Scientists and Engineers* (Boca Raton, FL: CRC Press)
[113] Hirota R 1977 Nonlinear partial difference equations: I. A difference analogue of the Korteweg–de Vries equation *J. Phys. Soc. Japan* **43** 1424–33
[114] Hirota R 1977 Nonlinear partial difference equations: II. Discrete-time Toda equation *J. Phys. Soc. Japan* **43** 2074–8
[115] Hirota R 1977 Nonlinear partial difference equations: III. Discrete sine-Gordon equation *J. Phys. Soc. Japan* **43** 2079–86
[116] Hirota R 1978 Nonlinear partial difference equations: IV. Bäcklund transformations for the discrete-time Toda equation *J. Phys. Soc. Japan* **45** 321–32
[117] Hirota R 1979 Nonlinear partial difference equations: V. Nonlinear equations reducible to linear equations *J. Phys. Soc. Japan* **46** 312–9
[118] Hydon P E 2000 *Symmetry Methods for Differential Equations* (Cambridge: Cambridge University Press)
[119] Hydon P E 2000 Symmetries and first integrals of ordinary difference equations *Roy. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci.* **456** 2835–55
[120] Hydon P E and Mansfield E L 2004 A variational complex for difference equations *Found. Comput. Math.* **4** 187–217
[121] Inönü E and Wigner E P 1953 On the contraction of groups and their representations *Proc. Natl. Acad. Sci. USA* **39** 510–24
[122] Ibragimov N H 1985 *Transformation Groups Applied to Mathematical Physics* (Boston: Reidel)
[123] Ibragimov N H (ed) 1993 *CRC Handbook of Lie Group Analysis of Differential Equations. Vol I: Symmetries, Exact Solutions and Conservation Laws* (Baton Rouge: CRC Press)
[124] Ibragimov N H (ed) 1994 *CRC Handbook of Lie Group Analysis of Differential Equations. Vol II: Applications in Engineering and Physicas Sciences* (Baton Rouge: CRC Press)
[125] Ibragimov N H (ed) 1995 *CRC Handbook of Lie Group Analysis of Differential Equations. Vol III* (Baton Rouge: CRC Press)
[126] Iserles A 2002 Brief introduction to Lie-group methods *Collected Lectures on the Preservation of Stability under Discretization* (Philadelphia: SIAM) pp 123–43
[127] Izmest'ev A A, Pogosyan G S, Sissakian A N and Winternitz P 1999 Contractions of Lie algebras and separation of variables. The n-dimensional sphere *J. Math. Phys.* **40** 1549–73
[128] Jacobson N 1962 *Lie Algebras* (New York: Interscience)
[129] Jiang Z–H 1998 Lie symmetries and their local determinancy for a class of differential-difference equations *Phys. Lett.* A **240** 137–43

[130] Jordan C 1939 *Calculus of Finite Differences* (Sopron: Röttig and Romwalter)
[131] Joshi N and Vassiliou P J 1995 The existence of Lie symmetries for first order analytic discrete dynamical systems *J. Math. Anal. Appl.* **195** 872–87
[132] Kersten P H M and Krasil'shchik I S 2000 *Symmetries and Recursion Operators for Classical and Supersymmetric Differential Equations* (Dordrecht: Kluwer)
[133] Kruskal M D, Joshi N and Halburd R 2004 Analytic and asymptotic methods for nonlinear singularity analysis: a review and extensions of tests for the Painleve property *Integrability of Nonlinear Systems* ed Y Kosmann-Schwarzbach, B Grammaticos and K M Tamizhmani (Berlin: Springer) pp 171–205
[134] Kumei S 1975 Invariance transformations, invariance group transformations, and invariance group of the sine-Gordon equations *J. Math. Phys.* **16** 2461–8
[135] Lafortune S, Martina L and Winternitz P 2000 Point symmetries of generalized Toda field theories *J. Phys. A: Math. Gen.* **33** 2419–35
[136] Lafortune S, Tremblay S and Winternitz P 2001 Symmetry classification of diatomic molecular chains *J. Math. Phys.* **42** 5341–57
[137] Lamb J S W and Quispel G R W 1995 Cyclic reversing k-symmetry groups *Nonlinearity* **8** 1005–26
[138] Lax P D 1968 Integrals of nonlinear equations of evolution and solitary waves *Commun. Pure Appl. Math.* **21** 467–90
[139] Levi D 1981 Nonlinear differential difference equations as Bäcklund transformations *J. Phys. A: Math. Gen.* **14** 1083–98
[140] Levi D 1992 Inhomogeneous Toda lattices and their symmetries *Group Theory in Physics* ed A Frank, T H Seligman and K B Wolf (New York: AIP) pp 97–108
[141] Levi D 1998 Symmetries and generalized symmetries for discrete dynamical systems *Symmetries and Integrability of Difference Equations* ed P A Clarkson and F W Nijhoff (Cambridge: Cambridge University Press) pp 261–74
[142] Levi D 2003 Lie symmetries for lattice equations *GROUP 24: Physical and Mathematical Aspects of Symmetries: Proc. 24th International Colloquium on Group Theoretical Methods in Physics (Paris, 15–20 July 2002)* ed J-P Gazeau, R Kerner, J-P Antoine, S Métens and J-Y Thibon (Bristol: Institute of Physics Publishing) pp 103–11
[143] Levi D and Benguria R 1980 Bäcklund transformations as nonlinear differential difference equations *Proc. Natl. Acad. Sci. USA* **77** 307–11
[144] Levi D and Martina L 2001 Integrable hierarchies of nonlinear difference-difference equations and symmetries *J. Phys. A: Math. Gen.* **34** 10357–68
[145] Levi D, Negro J and del Olmo M A 2001 Discrete derivatives and symmetries of difference equations *J. Phys. A: Math. Gen.* **34** 2023–30
[146] Levi D, Negro J and del Olmo M A 2001 Lie symmetries of difference equations *Czech. J. Phys.* **51** 341–8
[147] Levi D, Negro J and del Olmo M A 2004 Discrete q-derivatives and symmetries of q-difference equations *J. Phys. A: Math. Gen.* **37** 3459–73
[148] Levi D, Pilloni L and Santini P M 1981 Integrable three-dimensional lattices *J. Phys. A: Math. Gen.* **14** 1567–75
[149] Levi D and Ragnisco O 1979 Non-linear differential–difference equations with N-dependent coefficients II *J. Phys. A: Math. Gen.* **12** L163–7
[150] Levi D and Ragnisco O 1991 The inhomogeneous Toda lattice: its hierarchy and Darboux–Bäcklund transformations *J. Phys. A: Math. Gen.* **24** 1729–39
[151] Levi D and Ragnisco O (ed) 2000 *SIDE III—Symmetries and Integrability of Difference Equations (CRM Proc. and Lecture Notes vol 25)* (Providence, RI: American Mathematical Society)
[152] Levi D, Ragnisco O and Rodríguez M A 1992 On non-isospectral flows, Painlevé equations and symmetries of differential and difference equations *Teor. Mat. Fizika* **93** 473–80
[153] Levi D and Rodriguez M A 1992 Symmetry group of partial differential equations and of differential difference equations: the Toda lattice versus the Korteweg–de Vries equation *J. Phys. A: Math. Gen.* **25** L975–9
[154] Levi D and Rodríguez M A 1999 Lie symmetries for integrable evolution equations on the lattice *J. Phys. A: Math. Gen.* **32** 8303–16
[155] Levi D and Rodríguez M A 2004 Lie discrete symmetries of lattice equations *J. Phys. A: Math. Gen.* **37** 1711–25
[156] Levi D, Tempesta P and Winternitz P 2004 Umbral calculus, difference equations and the discrete Schrödinger equation *J. Math. Phys.* **45** 4077–105
[157] Levi D, Tempesta P and Winternitz P 2004 Lorentz and Galilei invariance on lattices *Phys. Rev.* D **69** 105011, 1–6
[158] Levi D, Tremblay S and Winternitz P 2000 Lie point symmetries of difference equations and lattices *J. Phys. A: Math. Gen.* **33** 8507–23

[159] Levi D, Tremblay S and Winternitz P 2001 Continuous symmetries of equations on lattices *Czech. J. Phys.* **51** 349–56

[160] Levi D, Tremblay S and Winternitz P 2001 Lie symmetries of multidimensional difference equations *J. Phys. A: Math. Gen.* **34** 9507–24

[161] Levi D, Vinet L and Winternitz P (ed) 1996 *Symmetries and Integrability of Difference Equations (CRM Proc. and Lecture Notes* vol 9*)* (Providence, RI: American Mathematical Society)

[162] Levi D, Vinet L and Winternitz P 1997 Lie group formalism for difference equations *J. Phys. A: Math. Gen.* **30** 663–49

[163] Levi D and Winternitz P 1988 The cylindrical Kadomtsev–Petviashvili equation: its Kac–Moody–Virasoro algebra and relation to KP equation *Phys. Lett.* A **129** 165–7

[164] Levi D and Winternitz P 1989 Nonclassical symmetry reduction: example of the Boussinesq equation *J. Phys. A: Math. Gen.* **22** 2915–24

[165] Levi D and Winternitz P 1991 Continuous symmetries of discrete equations *Phys. Lett.* A **152** 335–8

[166] Levi D and Winternitz P 1993 Symmetries of differential equations on a lattice. An example: the Toda lattice *Modern Group Analysis: Advanced Analytical and Computational Methods in Mathematical Physics* ed N H Ibragimov, M M Torrisi and A Valenti (Dordrecht: Kluwer) pp 265–72

[167] Levi D and Winternitz P 1993 Dimensional reduction for equations involving discrete and continuous variables *Applications of Analytic and Geometric Methods to Nonlinear Differential Equations* ed P A Clarkson (Dordrecht: Kluwer) pp 405–13

[168] Levi D and Winternitz P 1993 Symmetries and conditional symmetries of differential–difference equations *J. Math. Phys.* **34** 3713–30

[169] Levi D and Winternitz P 1995 Continuous symmetries of differential difference equations *Modern Group Theoretical Methods in Physics* ed J Bertrand, M Flato, J-P Gazeau, M Irac-Astaud and D Sternheimer (Dordrecht: Kluwer) pp 271–82

[170] Levi D and Winternitz P 1966 Symmetry analysis of discrete physical systems *Nonlinear Physics, Theory and Experiment* ed E Alfinito, M Boiti, L Martina and F Pempinelli (Singapore: World Scientific) pp 199–216

[171] Levi D and Winternitz P 1996 Lie point symmetries of differential difference equations *Symmetries and Integrability of Difference Equations* ed D Levi, L Vinet and P Winternitz (Providence, RI: American Mathematical Society) pp 199–208

[172] Levi D and Winternitz P 1996 Symmetries of discrete physical systems *Proc. 4th Wigner Symp.* ed N M Atakishiyev, T H Seligman and K B Wolf (Singapore: World Scientific) pp 382–91

[173] Levi D and Winternitz P 2002 Symmetries of discrete dynamical systems *J. Math. Phys.* **37** 5551–76

[174] Levi D and Winternitz P 2002 Lie point symmetries and commuting flows for equations on lattices *J. Phys. A: Math. Gen.* **35** 2249–62

[175] Levi D and Yamilov R 1997 Conditions for the existence of higher symmetries of evolutionary equations on a lattice *J. Math. Phys.* **38** 6648–74

[176] Levi D and Yamilov R 1999 Dilation symmetries and equations on the lattice *J. Phys. A: Math. Gen.* **32** 8317–23

[177] Levi D and Yamilov R 2000 Non-point integrable symmetries for equations on the lattice *J. Phys. A: Math. Gen.* **33** 4809–23

[178] Levi D and Yamilov R 2001 On the integrability of a new discrete nonlinear Schrödinger equation *J. Math. Phys.* **34** L553–62

[179] Y-s Li and G-c Zhu 1986 New set of symmetries of the integrable equations, Lie algebra and non-isospectral evolution equations: II. AKNS system *J. Phys. A: Math. Gen.* **19** 3713–25

[180] Lie S 1924 *Klassifikation und Integration von gewohnlichen Differentialgleichungen zwischen x, y die eine Gruppe von Transformationen gestatten. I–IV. Gessamelte Abhandlungen* vol 5 (Leipzig: Teubner)

[181] Lie S 1891 *Vorlesungen uber Differentialgleichungen mit bekannten infinitesimalen Transformationen, bearbeitet und herausgegeben von Dr. G. Scheffers* (Leipzig: Teubner)
Lie S 1967 *Differentialgleichungen* (New York: Chelsea) (reprinted)

[182] Lie S 1893 *Vorlesungen uber continuirliche Gruppen mit geometrischen und anderen Anwendungen, bearbeited und herausgegeben von Dr. G. Scheffers* (Leipzig: Teubner)

[183] Ma W-X and Fuchssteiner B 1999 Algebraic structure of discrete zero curvature equations and master symmetries of discrete evolution equations *J. Math. Phys.* **40** 2400–18

[184] Maeda S 1980 Canonical structure and symmetries for discrete systems *Math. Japan* **25** 405–20

[185] Maeda S 1981 Extension of discrete Noether theorem *Math. Japan* **26** 85–90

[186] Maeda S 1982 Lagrangian formulation of discrete systems and concept of difference space *Math. Japan* **27** 345–56

[187] Maeda S 1987 The similarity method for difference equations *IMA J. Appl. Math.* **38** 129–34

[188] Maeda S 1991 Application of numerical schemes admitting symmetries *Electron. Commun. Japan* 3 **73** 107–13
[189] Maillet J-M and Nijhoff F W 1989 Integrability for multidimensional lattice models *Phys. Lett.* B **224** 389–96
[190] Mahomed F M and Leach P G L 1989 Lie algebras associated with scalar second-order ordinary differential equations *J. Math. Phys.* **30** 2770–7
[191] Mahomed F M and Leach P G L 1989 The Lie algebra sl(3,R) and linearization *Quaest. Math.* **12** 121–39
[192] Manakov S V 1975 Complete integrability and stochastization of discrete dynamical systems *Sov. Phys.—JETP* **40** 269–74
[193] Martina L, Lafortune S and Winternitz P 2000 Point symmetries of generalized Toda field theories: II. Symmetry reduction *J. Phys. A: Math. Gen.* **33** 6431–46
[194] Martina L and Winternitz P 1989 Analysis and applications of the symmetry group of the multidimensional three-wave resonant interaction problem *Ann. Phys.* **196** 231–77
[195] Mikhailov A V 1979 Integrability of a two-dimensional generalization of the Toda chain *Sov. Phys. JETP Lett.* **30** 414–8
[196] Milne–Thomson L M 1933 *The Calculus of Finite Differences* (London: Macmillan)
[197] Moody R V and Patera J 1991 Discrete and continuous graded contractions of representations of Lie algebras *J. Phys. A: Math. Gen.* **24** 2227–58
[198] Naimark M A and Stern A I 1982 *Theory of Group Representations* (New York: Springer)
[199] Newell A C 1985 *Solitons in Mathematics and Physics* (Philadelphia: SIAM)
[200] Niederer U 1973 The maximal kinematical invariance group of the Schrödinger equation *Helv. Phys. Acta* **45** 802–10
[201] Nijhoff F W 1999 Discrete Painlevé equations and symmetry reduction on the lattice *Discrete Integrable Geometry and Physics* ed A Bobenko and R Seiler (New York: Oxford University Press) pp 209–34
[202] Nijhoff F W 1990 The direct linearisation approach to hierarchies of integrable PDEs in 2 + 1 dimensions: I. Lattice equations and the differential–difference hierarchies *Inverse Problems* **6** 567–90
[203] Nijhoff F and Capel H W 1995 The discrete Korteweg–de Vries equation *KdV '95* ed M Hazewinkel, H W Capel and E M de Jager (Dordrecht: Kluwer)
[204] Nijhoff F W, Papageorgiou V G and Capel H W 1992 Integrable time-discrete systems: lattices and mappings *Quantum Groups: Proc. Workshops, Euler Int. Math. Inst. (Leningrad, USSR, 1990) (Lecture Notes in Mathematics* vol 1510*)* pp 312–25
[205] Nijhoff F W, Papageorgiou V G, Capel H W and Quispel G R W 1992 The lattice Gelfand–Dikií hierarchy *Inverse Problems* **8** 597–621
[206] Nijhoff F W, Suris Yu and Viallet C (ed) 2003 SIDE V—symmetries and integrability of difference equations *J. Nonlinear Math. Phys.* **10** (Suppl. 2) 1–245
[207] Nimmo J J C and Schief W K 1997 Superposition principles associated with the Moutard transformation: an integrable discretization of a (2+1)-dimensional sine-Gordon system *Proc. R. Soc.* A **453** 255–79
[208] Noether E 1918 Invariante variationsprobleme *Nachr. Ges. Wiss. Goettingen, Math.-Phys. Kl.* **2** 235–57
See Noether E 1971 *Transp. Theory Stat. Phys.* **1** 186 (Engl. Transl.)
[209] Novikov S, Manakov S V, Pitaevski L P and Zakharov V E 1984 *Theory of Solitons. The Inverse Scattering Method* (New York: Consultants Bureau (Plenum)
[210] Ohta Y, Tamizhmani K M, Grammaticos B and Ramani A 1999 Singularity confinement and algebraic entropy: the case of the discrete Painlevé equations *Phys. Lett.* A **262** 152–7
[211] Olver P J 1993 *Applications of Lie Groups to Differential Equations* (New York: Springer)
[212] Olver P J 2003 Moving frames *J. Symb. Comp.* **36** 501–12
[213] Olver P J and Fels M 1998 Moving coframes: I. A practical algorithm *Acta Appl. Math.* **51** 161–213
[214] Olver P J and Fels M 1999 Moving coframes: II. Regularization and theoretical foundation *Acta Appl. Math.* **55** 127–208
[215] Olver P J and Rosenau Ph 1987 Group invariant solutions of differential equations *SIAM J. Appl. Math.* **47** 263–78
[216] Orfanidis S J 1978 Discrete sine-Gordon equations *Phys. Rev.* D **18** 3822–7
[217] Orlov A Yu and Winternitz P 1997 Algebra of pseudodifferential operators and symmetries of equations in the Kadomtsev–Petviashvili hierarchy *J. Math. Phys.* **38** 4644–74
[218] Ovsiannikov L V 1982 *Group Analysis of Differential Equations* (New York: Academic)
[219] Patera J, Sharp R T, Winternitz P and Zassenhaus H 1977 Continuous subgroups of the fundamental groups of physics: III. The de Sitter groups *J. Math. Phys.* **18** 2259–88
[220] Patera J, Winternitz P and Zassenhaus 1975 H continuous subgroups of the fundamental groups of physics: I. General method and the Poincaré group *J. Math. Phys.* **16** 1597–614
[221] Penskoi A V and Winternitz P 2004 Discrete matrix Riccati equations with superposition formulas *J. Math. Anal. Appl.* **294** 533–47

[222] Quispel G R W, Capel H W and Sahadevan R 1992 Continuous symmetries of differential difference equations; the Kac–van Moerbeke equation and Painlevé reduction *Phys. Lett.* A **170** 379–83

[223] Quispel G R W and Sahadevan R 1993 Lie symmetries and integration of difference equations *Phys. Lett.* A **184** 64–70

[224] Ramani A and Grammaticos B 2002 What is the discrete analogue of the Painlevé property? *ANZIAM J.* **44** 21–32

[225] Ramani A, Grammaticos B and Tamizhmani K M 1993 Painlevé analysis and singularity confinement: the ultimate conjecture *J. Phys. A: Math. Gen.* **26** L53–8

[226] Ramani A, Grammaticos B, Tamizhmani T and Tamizhmani K M 2003 The road to the discrete analogue of the Painlevé property: Nevanlinna meets singularity confinement *Comput. Math. Appl.* **45** 1001–12

[227] Rand D W, Winternitz P and Zassenhaus H 1988 On the identification of a Lie algebra given by its structure constants: I. Direct decompositions, Levi decomposition and nilradicals *Lin. Algebra Appl.* **109** 197–246

[228] Remoissenet M 1999 *Waves Called Solitons. Concepts and Experiments* (Berlin: Springer)

[229] Rodríguez M A and Winternitz P 2004 Lie symmetries and exact solutions of first-order difference schemes *J. Phys. A: Math. Gen.* **37** 6129–42

[230] Rogers C and Ames W F 1989 *Nonlinear Boundary Value Problems in Science and Engineering* (Boston, MA: Academic)

[231] Rogers C, Schief W K and Winternitz P 1997 Lie-theoretical generalization and discretization of the Pinney equation *J. Math. Anal. Appl.* **216** 246–64

[232] Roman S 1984 *The Umbral Calculus* (San Diego: Academic)

[233] Rota G C 1975 *Finite Operator Calculus* (San Diego: Academic)

[234] Rota G C 2003 *Gian-Carlo Rota on Analysis and Probability. Selected Papers and Commentaries* ed J Dhombres, J P S Kung and N Starr (Boston, MA: Birkhäuser)

[235] Sahadevan R 2001 Nonlinear differential-difference and difference equations: integrability and exact solvability *Comp. Math. Appl.* **42** 627–37

[236] Sahadevan R, Byrnes G B and Quispel G R W 1994 Linearisation of difference equations using factorisable Lie symmetries *Proc. Workshop on Symmetries and Integrability of Difference Equation 1994 (Esterel Montreal, Canada)* ed D Levi, L Vinet and P Wintemitz (Providence, RI: American Mathematical Society) pp 337–43

[237] Sahadevan R and Khousalya S 2000 Similarity reduction of a $(2+1)$ Volterra *J. Phys. A: Math. Gen.* **33** L171–6

[238] Sahadevan R and Quispel G R W 1997 Lie symmetries and linearization of the QRT mapping *Physica* A **234** 775–84

[239] Saito S, Saitoh N, Yamamoto J and Yoshida K 2001 A characterization of discrete time soliton equations *J. Phys. Soc. Japan* **70** 3517–23

[240] Hietarinta J, Nijhoff F W and Satsuma J (ed) 2001 SIDE IV—symmetries and integrability of difference equations *J. Phys. A: Math. Gen.* **34** 10337–774

[241] Schief W K 1997 A discrete Pinney equation *Appl. Math. Lett.* **10** 13–5

[242] Schief W K and Rogers C 2000 Bäcklund transformations and the integrable discretization of characteristic equations *Chaos Solitons Fractals* **11** 107–13

[243] Schwarz F 1982 A REDUCE package for determining Lie symmetries of ordinary and partial differential equations *Comput. Phys. Commun.* **27** 179–86

[244] Shabat A B 2003 Universal models of soliton hierarchies *Theor. Math. Phys.* **136** 1066–76

[245] Shabat A B and Yamilov R I 1991 Symmetries of nonlinear chains *Leningrad Math. J.* **2** 377–400

[246] Sokolov V V and Shabat A B 1984 Classification of integrable evolution equations *Sov. Sci. Rev. Sect. C Math. Phys. Rev.* **4** 221–80

[247] Steeb W-H 1996 *Continuous Symmetries, Lie Algebras, Differential Equations and Computer Algebra* (Singapore: World Scientific)

[248] Stephani H 1990 *Differential Equations: Their Solution Using Symmetries* ed M MacCallum (Cambridge: Cambridge University Press)

[249] Stormark O 2000 *Lie's Structural Approach to PDE Systems* (Cambridge: Cambridge University Press)

[250] Suris and Yu B 2003 *The Problem of Integrable Discretization: Hamiltonian Approach* (Basel: Birkhäuser)

[251] Tamizhmani K M, Ramani A, Grammaticos B and Ohta Y 1999 Integrability criteria for differential–difference systems: a comparison of singularity confinement and low-growth requirements *J. Phys. A: Math. Gen.* **32** 6679–85

[252] Toda T 1967 Vibration of a chain with nonlinear interaction *J. Phys. Soc. Japan* **22** 431–6

[253] Toda M 1981 *Theory of Nonlinear Lattices* (Berlin: Springer)

[254] Toda M 1989 *Nonlinear Waves and Solitons* (Dordrecht: Kluwer)

[255] Thomas J W 1995 *Numerical Partial Differential Equations: Finite Difference Methods* (New York: Springer)

[256] Tremblay S, Grammaticos B and Ramani A 2001 Integrable lattice equations and their growth properties *Phys. Lett.* A **278** 319–24

[257] Turbiner A and Winternitz P 1999 Solutions of nonlinear differential and difference equations with superposition formulas *Lett. Math. Phys.* **50** 189–201

[258] Tzitzeica G 1907 Sur une nouvelle classes de surfaces *C. R. Acad. Sci. Paris* **144** 1257–9

[259] Tzitzeica G 1910 Sur une nouvelle classe de surfaces *C. R. Acad. Sci. Paris* **150** 955–6

[260] Valiquette F and Winternitz P 2005 Discretization of partial differential equations preserving their point symmetries *J. Phys. A: Math. Gen.* **38** 9765–83

[261] Winternitz P 1988 Kac–Moody–Virasoro symmetries of integrable nonlinear partial differential equations *Symmetries and Nonlinear Phenomena (Paipa, 1988)* ed D Levi and P Winternitz (Teaneck: World Scientific) pp 358–75

[262] Winternitz P 1993 Lie Groups and solutions of nonlinear partial differential equations *Integrable Systems, Quantum Groups and Quantum Fields Theories* ed A Ibort and M A Rodriguez (Dordrecht: Kluwer) pp 429–95

[263] Winternitz P 1999 Lie groups and difference equations *Group 22, Proc. 22nd Int. Colloquium on Group Theoretical Methods in Physics* ed S P Corney, R Delbourgo and P D Jarvis (Boston, MA: International)

[264] Winternitz P 2004 Symmetries of discrete systems *Discrete Integrable Systems* ed B Grammaticos, Y Kosmann-Schwarzbach and T Tamizhmani (Berlin: Springer) pp 185–243

[265] Winternitz P 2005 Symmetries of linear difference systems *Group Theory Methods in Physics* ed G S Pogosyan, L E Vicent and K B Wolf (Bristol, UK: IOP) pp 121–9

[266] Winternitz P, Gomez-Ullate D, Iserles A, Levi D, Olver P J, Quispel R and Tempesta P (eds) 2005 *Group Theory and Numerical Analysis* (CRM Proc. Lect. Notes 39) (Providence, RI: American Mathematical Society)

[267] Yamilov R I 2004 Relativistic Toda chains and Schlesinger transformations *Theor. Math. Phys.* **139** 623–35

[268] Yamilov R and Levi D 2004 Integrability conditions for n and t dependent dynamical lattice equations *J. Nonlinear Math. Phys.* **11** 75–101

[269] Zakharov V E (ed) 1991 *What is Integrability?* (New York: Springer)